Tufts University

Math 42 | **Department of Mathematics** | **November 10, 2014**

Exam 2 | **12:00 pm to 1:20 pm**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) If $\nabla f(x,y,z) \neq \mathbf{0}$, the direction that yields the largest directional derivative of f at (x,y,z) is $\frac{\nabla f(x,y,z)}{|\nabla f(x,y,z)|}$. **Answer:** $\boxed{\text{True}}$

   (b) If $D = \{(x,y) \mid h_1(y) \leq x \leq h_2(y),\, c \leq y \leq d\}$, then the area of D is given by $\displaystyle\int_{h_1(y)}^{h_2(y)} \int_c^d 1\, dy\, dx$.
   Answer: $\boxed{\text{False}}$ (order of integration does not make sense).

   (c) Suppose D is an open simply-connected region containing the unit circle, C. Let $F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, such that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ for all points in D, then $\displaystyle\int_C \mathbf{F}(x,y) \cdot d\mathbf{r} = 0$.
   Answer: $\boxed{\text{True}}$

   (d) The vector field $\mathbf{F}(x,y,z) = \langle x^2, y, z\rangle$ is conservative on $\mathbb{R}^3$.
   Answer: $\boxed{\text{True}}$ ($\frac{\partial(y)}{\partial x} = 0 = \frac{\partial(x^2)}{\partial y}$, $\frac{\partial(x^2)}{\partial z} = 0 = \frac{\partial(z)}{\partial x}$, $\frac{\partial(y)}{\partial z} = 0 = \frac{\partial(z)}{\partial y}$).

   (e) $f(x,y,z) = xz + x^2y + \sin(y)$ is a potential function for the vector field, $\mathbf{F}(x,y,z) = \langle 2xy + z, x^2 + \cos(y), x\rangle$.
   Answer: $\boxed{\text{True}}$ ($\nabla f = \langle z + 2xy, x^2 + \cos(y), x\rangle = \mathbf{F}(x,y,z)$).

2. (10 points) Suppose that $z = \sin(\pi x)\cos(\pi y)$, and $x = se^t$, $y = s + t$. Find $\frac{\partial z}{\partial t}$ when $s = 1$ and $t = 0$.
   Answer:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (\pi\cos(\pi x)\cos(\pi y))(se^t) + (-\pi\sin(\pi x)\sin(\pi y))(1)$$

$$x(1,0) = 1, \qquad y(1,0) = 1$$

$$\Rightarrow \frac{\partial z}{\partial t} = \pi(-1)(-1)(1) + 0 = \boxed{\pi}.$$

3. (10 points) Let $f(x,y) = 9x - 3x^3 - 12y + y^3$.

(a) Find all critical points of $f(x,y)$ in the xy–plane.
Answer: The critical points occur when the partial derivatives are 0 or undefined.

$$f_x = 9 - 9x^2 \text{ and } f_y = 3y^2 - 12.$$

These are never undefined and:

$$\begin{aligned} f_x &= 0 & \text{and} \qquad f_y &= 0 \\ 9 - 9x^2 &= 0 & 3y^2 - 12 &= 0 \\ x^2 &= 1 & y^2 &= 4 \\ x &= \pm 1 & y &= \pm 2 \end{aligned}$$

Thus, the critical points are: $\boxed{(1,2)}$, $\boxed{(-1,2)}$, $\boxed{(1,-2)}$, and $\boxed{(-1,-2)}$.

(b) Determine where $f(x,y)$ has a local maximum, local minimum, or a saddle point. Determine the values of f at these points as well.
Answer: Here, we use the second derivative test:

$$f_{xx} = -18x, \quad f_{yy} = 6y, \quad f_{xy} = 0, \quad D = (-18x)(6y) - 0^2 = -108xy.$$

$\boxed{D(1,2) < 0 \Rightarrow (1,2) \text{ is a saddle with } f(1,2) = -10.}$

$\boxed{D(-1,2) > 0, f_{xx}(-1,2) > 0 \Rightarrow (-1,2) \text{ is a local min with } f(-1,2) = -22.}$

$\boxed{D(1,-2) > 0, f_{xx} < 0 \Rightarrow (1,-2) \text{ is a local max with } f(1,-2) = 22.}$

$\boxed{D(-1,-2) < 0 \Rightarrow (-1,-2) \text{ is a saddle with } f(-1,-2) = 10.}$

4. (10 points) Using Lagrange multipliers, find the absolute maximum and minimum values of

$$f(x, y, z) = 2x + 4y - 12z,$$

on the ellipse $x^2 + 2y^2 + z^2 = 39$.
Answer: We solve

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= 39. \end{aligned}$$

Then, we have

$$\begin{aligned} 2 &= 2\lambda x, \\ 4 &= 4\lambda y, \\ -12 &= 2\lambda z, \\ x^2 + 2y^2 + z^2 &= 39. \end{aligned}$$

Consider the first equation, $1 = \lambda x \Rightarrow$ neither λ nor x can be zero. Thus,

$$\lambda = \frac{1}{x}.$$

$$1 = \frac{y}{x} \Rightarrow y = x.$$

$$-6 = \frac{z}{x} \Rightarrow z = -6x.$$

$$\Rightarrow x^2 + 2(x)^2 + (-6x)^2 = 39x^2 = 39$$

$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

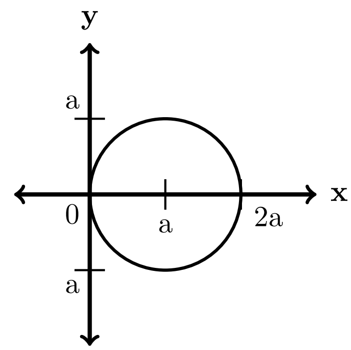
$$f(1, 1, -6) = 78 \qquad f(-1, -1, 6) = -78.$$

Thus, the $\boxed{\text{absolute maximum occurs at } (1, 1, -6) \text{ with value } f(x, y, z) = 78}$ and the $\boxed{\text{absolute minimum occurs at } (-1, -1, 6) \text{ with value } f(x, y, z) = -78}$.

The exam continues on the back!

5. (10 points) The area inside the circle $(x-a)^2 + y^2 = a^2$ is πa^2, since this is a circle of radius a.

(a) Sketch the circle and, using inequalities, express the region, D, inside the circle in **both** Cartesian **and** polar coordinates.
Answer:



Cartesian Coordinates:

$$\boxed{D = \{(x,y) \in \mathbb{R}^2 |\ -\sqrt{a^2-(x-a)^2} \le y \le \sqrt{a^2-(x-a)^2},\ 0 \le x \le 2a\}}.$$

Polar Coordinates:

$$(x-a)^2 + y^2 - a^2 = x^2 - 2ax + a^2 + y^2 - a^2 = r^2 - 2ar\cos(\theta) = 0.$$

$$\Rightarrow r = 2a\cos(\theta).$$

$$\Rightarrow \boxed{D = \{(r,\theta) \in \mathbb{R}^2 |\ 0 \le r \le 2a\cos(\theta),\ \ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\}}.$$

(b) Set up, **but DO NOT EVALUATE**, the integral for computing the area, $\iint_D 1\,dA$, in Cartesian coordinates. **Answer:**

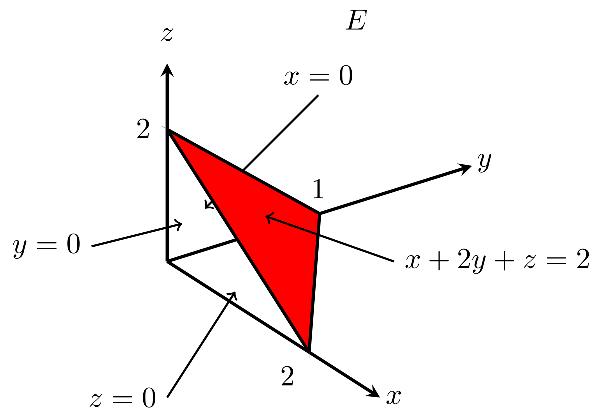
$$\boxed{\int_0^{2a} \int_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}} dy\,dx}.$$

(c) Set up, **but DO NOT EVALUATE**, the integral for computing the area in polar coordinates. **Answer:**

$$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos(\theta)} r\,dr\,d\theta}.$$

6. (15 points) Consider the solid region that is bounded by the 4 planes, $x = 0$, $y = 0$, $z = 0$, and $x + 2y + z = 2$. Denote this region by E.

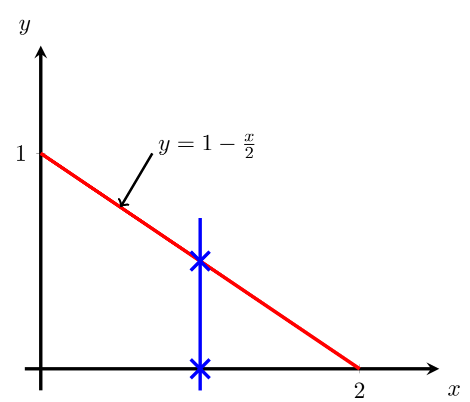
   a. Draw the solid region, E, clearly labeling the bounding planes and any intersection points with the coordinate axes.
   Answer:



   b. Set up, **but DO NOT EVALUATE**, the triple integral of $f(x, y, z) = z$ over the region V using the following order of integrations:

      i. $dzdydx$
      Answer:

      Slicing in z first yields: $0 \leq z \leq 2 - x - 2y$. The 2D projection in the $xy-$plane is:



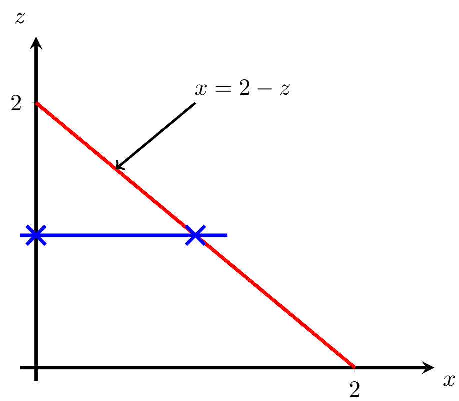
$$0 \leq y \leq 1 - \frac{x}{2},$$
$$0 \leq x \leq 2.$$

$$\boxed{\iiint_E z dV = \int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{2-x-2y} z\, dz\, dy\, dx}.$$

ii. $dydxdz$

Answer:

Slicing in y first yields: $0 \leq y \leq 1 - \frac{x}{2} - \frac{z}{2}$. The 2D projection in the $xz-$plane is:



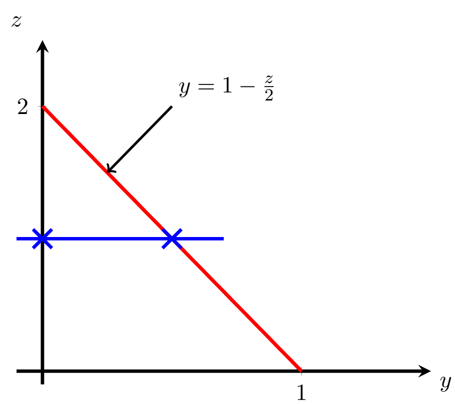
$$0 \leq x \leq 2 - z,$$
$$0 \leq z \leq 2.$$

$$\boxed{\iiint\limits_E z dV = \int_0^2 \int_0^{2-z} \int_0^{1-\frac{x}{2}-\frac{z}{2}} z \, dy \, dx \, dz}.$$

iii. $dxdydz$

Answer:

Slicing in x first yields: $0 \leq x \leq 2 - 2y - z$. The 2D projection in the $yz-$plane is:



$$0 \leq y \leq 1 - \frac{z}{2},$$
$$0 \leq z \leq 2.$$

$$\boxed{\iiint\limits_E z dV = \int_0^2 \int_0^{1-\frac{z}{2}} \int_0^{2-2y-z} z \, dx \, dy \, dz}.$$

7. (15 points) Consider the region bounded above by the sphere, $x^2 + y^2 + z^2 = 3$, and below by the cone, $z = \sqrt{3(x^2 + y^2)}$.

(a) Set up but **DO NOT EVALUATE** the triple integral representing the volume of the region using cylindrical coordinates (**Hint:** The intersection of the sphere and the cone occurs in the plane $z = \frac{3}{2}$).
Answer:
In cylindrical coordinates, the sphere is written as $r^2 + z^2 = 3$ and the cone is $z = \sqrt{3}r$. Thus,

$$\sqrt{3}r \leq z \leq \sqrt{3 - r^2}.$$

Since the intersection of the cone and sphere is at $z = \frac{3}{2}$, the projection on the xy-plane is the circle centered at $(0, 0)$ with radius:

$$\frac{3}{2} = \sqrt{3}r \Rightarrow r = \frac{\sqrt{3}}{2}.$$

Then, the integral becomes:

$$\boxed{V = \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \int_{\sqrt{3}r}^{\sqrt{3-r^2}} r\, dz\, dr\, d\theta}.$$

(b) Evaluate the triple integral using spherical coordinates.
Answer:
In spherical coordinates, the sphere is $\rho = \sqrt{3}$ and the cone is determined by:

$$\rho\cos(\theta) = \sqrt{3}\rho\sin(\theta) \Rightarrow \tan(\theta) = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}.$$

Thus, the integral becomes

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^{\sqrt{3}} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \frac{\rho^3}{3} \sin(\theta)\Big|_0^{\sqrt{3}}\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \sqrt{3}\sin(\theta)\, d\phi\, d\theta = \sqrt{3} \int_0^{2\pi} (-\cos(\theta))\Big|_0^{\frac{\pi}{6}}\, d\theta$$

$$= \sqrt{3} \int_0^{2\pi} \left(-\frac{\sqrt{3}}{2} - -1\right) d\theta = \sqrt{3}\left(1 - \frac{\sqrt{3}}{2}\right) 2\pi = \boxed{\sqrt{3}\pi\left(2 - \sqrt{3}\right)}.$$

8. (10 points) Consider the line segment, C, joining the origin and the point $(1, 1, 1)$. Let $f(x, y, z) = x - 3y^2 + z$.

(a) Integrate $f(x, y, z)$ over C, $\int_C f(x, y, z)\, ds$.

Answer:
We can easily parametrize the line segment by:

$$\mathbf{r}(t) = \langle t, t, t \rangle \text{ for } 0 \le t \le 1.$$

Then, $\mathbf{r}'(t) = \langle 1, 1, 1 \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{1 + 1 + 1} = \sqrt{3}$.

$$\Rightarrow \int_C f(x, y, z)\, ds = \int_0^1 f(\mathbf{r}(t))\sqrt{3}\, dt$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt = \sqrt{3} \int_0^1 2t - 3t^2\, dt = \sqrt{3}\left(t^2 - t^3\right)\Big|_0^1 = \boxed{0}.$$

(b) Find the line integral of the gradient field, ∇f, over the same curve, C, $\int_C \nabla f \cdot d\mathbf{r}$.

Answer:
By the fundamental theorem of line integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(1, 1, 1) - f(0, 0, 0) = (1 - 3 + 1) - (0 - 0 + 0) = \boxed{-1}.$$

End of Exam