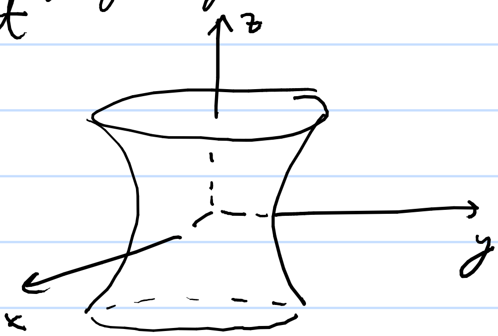
Math 42 Fall 2014
Exam 1 Solutions

1. (a) False

(b) True: the line has direction vector $\vec{v} = \langle 1, -2, 1\rangle$ and the plane has normal $\vec{n} = \langle 1, 2, 3\rangle$.
Since $\vec{n} \cdot \vec{v} = 0$, the line is parallel to the plane.

(c) False The graph of $x^2 + y^2 = z^2 + 1$ is a hyperboloid of one sheet



(d) False $\vec{r}\,'(t) = -\frac{2}{3}\sin t\,\vec{i} - \frac{2}{3}\cos t\,\vec{j} - \frac{1}{3}\vec{k}$,

so

$$\|\vec{r}\,'(t)\| = \sqrt{\left(-\frac{2}{3}\sin t\right)^2 + \left(-\frac{2}{3}\cos t\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{1}{9}} = \frac{\sqrt{5}}{9} \neq 1.$$

(e) True: Setting $f(x,y) = 1$ gives

$$x^2 - y^2 + 1 = 1 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y.$$

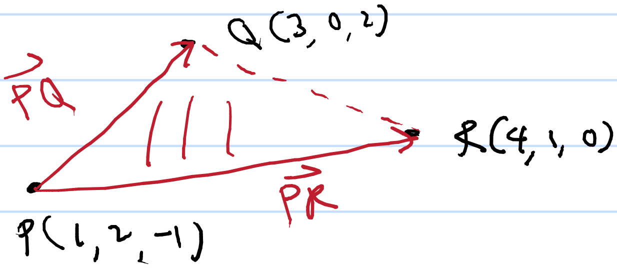
2. (a) $\text{scal}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \dfrac{(2\vec{i}+3\vec{j}+3\vec{k})\cdot(\vec{i}-2\vec{j}+2\vec{k})}{\sqrt{1^2+(-2)^2+(2)^2}}$

$$= \frac{2}{3}.$$

(b) $\text{proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$

$$= \frac{2}{9}\left(\vec{i}-2\vec{j}+2\vec{k}\right).$$

3



(a) $\text{Area}(\Delta PQR) = \frac{1}{2}|\vec{PQ}\times\vec{PR}|.$

Now $\vec{PQ} = 2\vec{i}-2\vec{j}+3\vec{k}$ and $\vec{PR} = 3\vec{i}-\vec{j}+\vec{k}$,

so $\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 3 \\ 3 & -1 & 1 \end{vmatrix} = \vec{i}+7\vec{j}+4\vec{k}.$

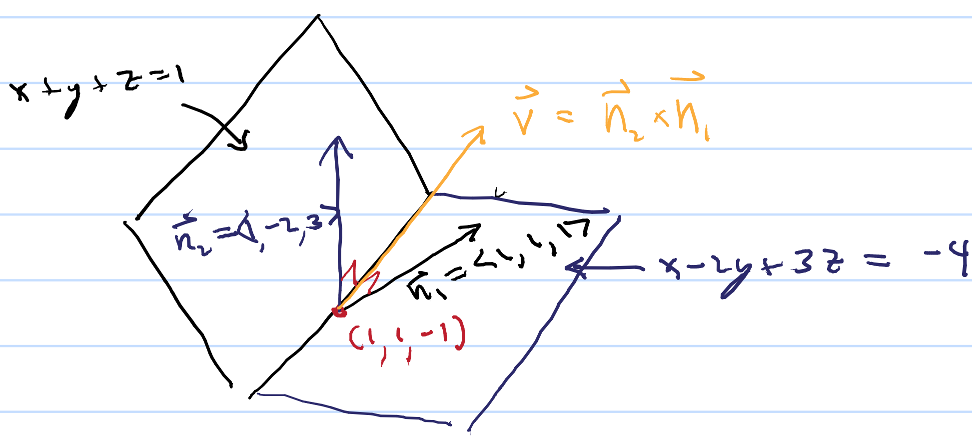
Hence $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{1^2+7^2+4^2}$
$$= \sqrt{66}/2.$$

(b) The plane containing P, Q, and R has normal $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \vec{i} + 7\vec{j} + 4\vec{k}$. Using P as a point on this plane, we conclude that it has equation

$$1(x-1) + 7(y-2) + 4(z+1) = 0, \text{ or}$$

$$x + 7y + 4z = 11.$$

4



The line of intersection has direction vector

$$\vec{v} = \vec{n}_2 \times \vec{n}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -5\vec{i} + 2\vec{j} + 3\vec{k}.$$

Since the line goes through $(1,1,-1)$, its parametric equations are

$$x = 1-5t, \quad y = 1+2t, \quad z = -1+3t.$$

5 (a) $\vec{v}(t) = \int \vec{a}(t)\,dt$

$$= \int -10\vec{k}\,dt$$
$$= -10t\vec{k} + \vec{C}$$

Now

$\vec{v}(0) = \vec{C} = \vec{i} + \vec{j} - \vec{k}$, so

$$\vec{v}(t) = -10t\vec{k} + \vec{i} + \vec{j} - \vec{k} = \vec{i} + \vec{j} - (10t+1)\vec{k}.$$

(b) $\vec{r}(t) = \int \vec{v}(t)\,dt = \int (\vec{i} + \vec{j} - (10t+1)\vec{k})\,dt$

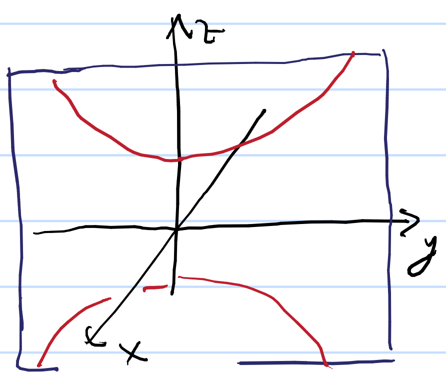
$$= t\vec{i} + t\vec{j} - (5t^2 + t)\vec{k} + \vec{D}$$

Since

$\vec{r}(0) = 2\vec{i} + 3\vec{j} = \vec{D}$, we have

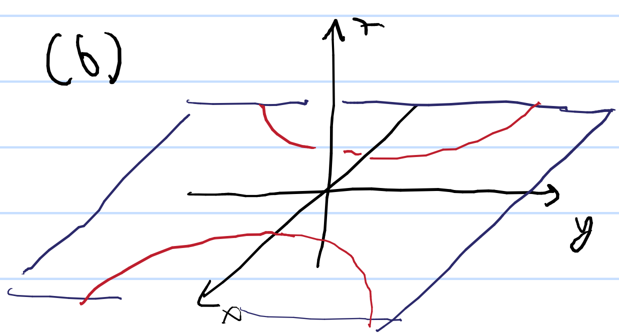
$$\vec{r}(t) = (t+2)\vec{i} + (t+3)\vec{j} - (5t^2 + t)\vec{k}.$$

6 (a)



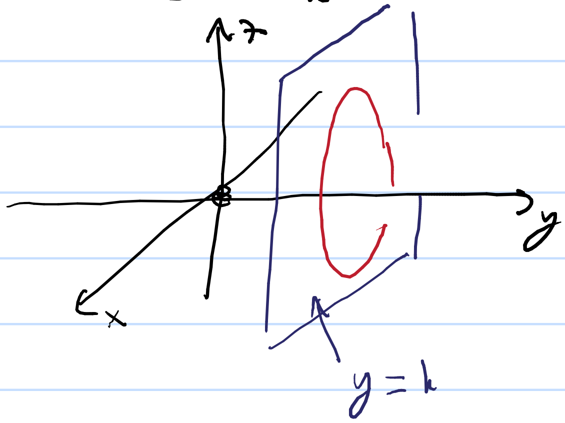
The yz-trace is the hyperbola $z^2 - y^2 = 1$

(b)

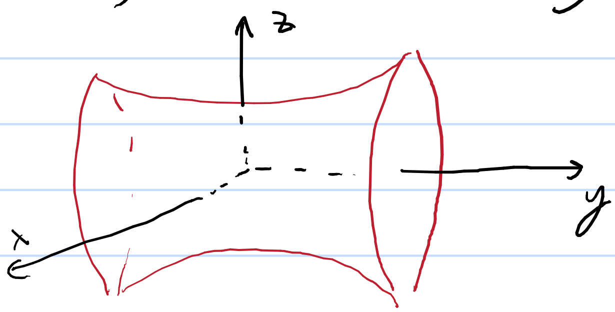


The xy-trace is the hyperbola $x^2 - y^2 = 1$.

(c) The trace in any plane $y=k$ is circle
$x^2+z^2 = k^2+1$.

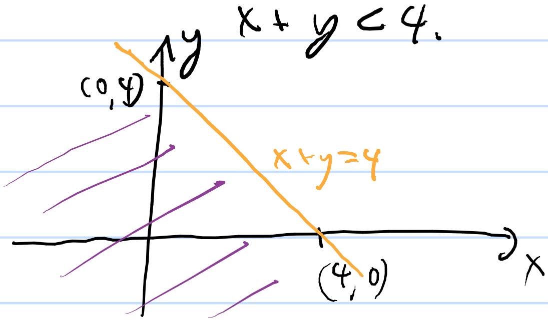


(d) $x^2-y^2+z^2=1$ is the equation of a hyperboloid of one sheet, with axis along the y-axis.



(7) $f(x,y) = \ln(4-x-y)$.
(a) The domain of $f(x,y)$ is the set of all points (x,y) in the plane such that $4-x-y > 0$, or equivalently,

$x+y<4$.



(b) It is the half plane below the line $x+y=4$ but not including the line.

(c) $f(x,y) = 0 \iff \ln(4-x-y) = 0$

$\iff 4-x-y = 1$

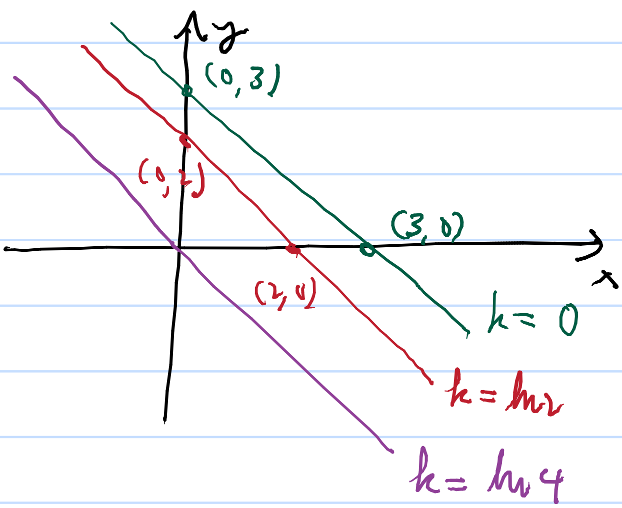
$\iff x+y = 3.$

$f(x,y) = \ln 2 \iff \ln(4-x-y) = \ln 2$

$\iff 4-x-y = 2$

$\iff x+y = 2.$

$f(x,y) = \ln 4 \iff x+y = 0.$



(8) $f(x,y) = e^x(\cos y + \sin y)$

(a)

$$\frac{\partial f}{\partial x} = e^x(\cos y + \sin y) \; ; \; \frac{\partial f}{\partial y} = e^x(-\sin y + \cos y)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = e^x(\cos y + \sin y) \; ; \; \frac{\partial^2 f}{\partial y \partial x} = e^x(-\sin y + \cos y) \; ;$$

$$\frac{\partial^2 f}{\partial y^2} = e^x(-\cos y - \sin y).$$

(b) From (a),

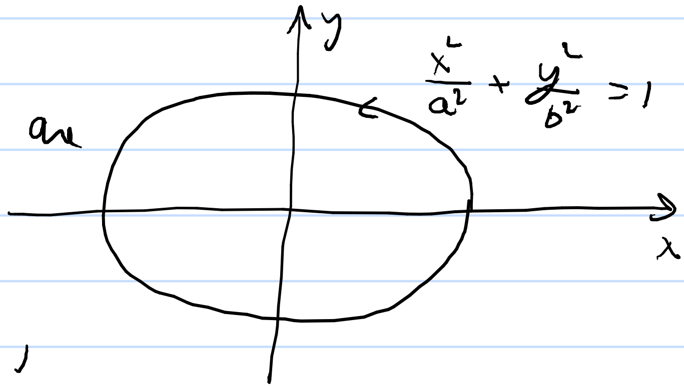
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x(\cos y + \sin y) + e^x(-\cos y - \sin y)$$

$$= 0.$$

(9) The parametric equations $x = a\cos t$, $b\sin t$ $(0 \le t \le 2\pi)$ are equivalent to the vector function

$$\vec{r}(t) = a\cos t\,\vec{i} + b\sin t\,\vec{j},$$

for $0 \le t \le 2\pi$.



Now $\vec{r}'(t) = -a\sin t\,\vec{i} + b\cos t\,\vec{j}$. Hence the arc length of the ellipse is

$$L = \int_0^{2\pi} |\vec{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (b\cos t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt.$$