Math 42 **Final Examination Solutions** Fall, 2012

1. (15 points) *For each item below,* **box in** *your answer. In (b) and (c), we will look* only *at your* *boxed-in answer*.

   (a) *Calculate the vector projection* ($\mathrm{proj}_{\vec{w}}\,\vec{v}$) *of* $\vec{v}$ *in the direction of* $\vec{w}$*, where*

$$\vec{v} = \vec{\imath} + \vec{\jmath} + \vec{k}$$
$$\vec{w} = \vec{\imath} - 2\vec{\jmath} - \vec{k} :$$

$$\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}} = \frac{1-2+1}{1+4+1}\vec{w} = -\frac{1}{3}\vec{w} =$$

$\boxed{-\frac{1}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{2}{3}\vec{k}}$.

   (b) *Rewrite the iterated integral*

$$\int_0^2 \int_{\sqrt{x/2}}^1 (\sin x)\sqrt{1-y}\,dy\,dx$$

with the order of integration reversed:
The region of integration is given by

$$\begin{cases} \sqrt{\frac{x}{2}} \le y \le 1 \\ 0 \le x \le 2 \end{cases}.$$

This is the region bounded by the y-axis, the line $y = 1$ and the parabola $y = \sqrt{x/2}$ from the origin to $(2, 1)$. Slicing this horizontally instead of vertically, we get the description

$$\begin{cases} 0 \le x \le 2y^2 \\ 0 \le y \le 1 \end{cases}$$

leading to the iterated integral $\boxed{\int_0^1 \int_0^{2y^2} (\sin x)\sqrt{1-y}\,dx\,dy}$.

   (c) *Set up (but do not attempt to evaluate) an integral expressing the length of the curve* $\mathcal{C}$ *given parametrically by*

$$x = 2\sin t$$
$$y = 3\cos t \qquad 0 \le t \le 2\pi$$
$$z = \frac{t}{\pi} :$$

From the parametrization, we get

$$dx = 2\cos t\,dt$$
$$dy = -3\sin t\,dt$$
$$dz = \frac{1}{\pi}\,dt$$

so the element of arc length is

$$\begin{aligned} ds &= \sqrt{(2\cos t)^2 + (-3\sin t)^2 + \left(\frac{1}{\pi}\right)^2}\, dt \\ &= \sqrt{4 + \frac{1}{\pi^2} + 5\sin^2 t}\, dt \end{aligned}$$

and the arc length is $\boxed{\int_0^{2\pi} \sqrt{4 + \frac{1}{\pi^2} + 5\sin^2 t}\, dt.}$

2. (10 points) *Find an equation for the plane containing the three points* $P(1, 0, -1)$, $Q(2, -1, 0)$, and $R(3, 2, -1)$:

$$\begin{aligned} \vec{PQ} &= \vec{\imath} - \vec{\jmath} + \vec{k} \\ \vec{PR} &= 2\vec{\imath} + 2\vec{\jmath} \\ \vec{n} = \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{vmatrix} \\ &= -2\vec{\imath} + 2\vec{\jmath} + 4\vec{k} \end{aligned}$$

so the equation of the plane is

$$-2(x-1) + 2y + 4(z+1) = 0$$

or

$$-2x + 2y + 4z = -6.$$

3. (10 points) *Find the absolute maximum and absolute minimum values of the function*

$$f(x, y, z) = x + 2y - 2z$$

on the sphere

$$x^2 + y^2 + z^2 = 9:$$

In theory, you might be able to do this using spherical coordinates, but it is much more efficient (and easier) to use Lagrange multipliers. The Lagrange multiplier condition $\vec{\nabla} f = \lambda \vec{\nabla} g$ reads

$$\begin{aligned} 1 &= \lambda(2x) \\ 2 &= \lambda(2y) \\ -2 &= \lambda(2z); \end{aligned}$$

since these imply that $\lambda \neq 0$, we can write

$$\begin{aligned} x &= \frac{1}{2\lambda} \\ y &= \frac{2}{2\lambda} \\ z &= -\frac{2}{2\lambda} \end{aligned}$$

or

$$\begin{aligned} y &= 2x \\ z &= -2x; \end{aligned}$$

substituting this into the equation of the surface

$$\begin{aligned} x^2 + (2x)^2 + (-2x)^2 &= 9 \\ 9x^2 &= 9 \\ x^2 &= 1 \\ x &= \pm 1. \end{aligned}$$

This yields two points, and the corresponding values of f are

$$\begin{aligned} f(1, 2, -2) &= 9 \\ f(-1, -2, 2) &= -9. \end{aligned}$$

The first is the maximum, the second is the minimum:

$$\begin{aligned} \max_{x^2+y^2+z^2=9} (x + 2y - 2z) &= f(1, 2, -2) = 9 \\ \min_{x^2+y^2+z^2=9} (x + 2y - 2z) &= f(-1, -2, 2) = -9. \end{aligned}$$

4. (10 points) *The function*

$$f(x, y) = \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} - 2xy + \frac{y^2}{2}$$

has critical points at $(0, 0)$, $(1, 2)$ *and* $(-3, -6)$. *Analyze each of the* two *critical points* $(0, 0)$ *and* $(1, 2)$ *to determine whether it is a local minimum, a local maximum, or neither (i.e., a saddle point). (*Don't bother with the third critical point.*)*:

The partials of the function are

$$\begin{aligned} f_x &= x^3 + 2x^2 + x - 2y \\ f_y &= -2x + y \\ f_{xx} &= 3x^2 + 4x + 1 \\ f_{xy} = f_{yx} &= -2 \\ f_{yy} &= 1 \end{aligned}$$

so the discriminant is

$$\Delta(x,y) = \begin{vmatrix} 3x^2+4x+1 & -2 \\ -2 & 1 \end{vmatrix} = 3x^2+4x-3.$$

The values at the chosen critical points are

- $\Delta(0,0) = -3 < 0$ so this is a saddle point;
- $\Delta(1,2) = 4 > 0$ and $f_{xx} = 8 > 0$ so this point is a local minimum.

5. (10 points) *Consider the vector field*

$$\vec{F}(x,y,z) = (x^2 - z^2)\vec{\imath} + (y^2 - x^2)\vec{\jmath} + (z^2 - y^2)\vec{k}.$$

(a) *Calculate the divergence of* $\vec{F}$, $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$.

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = 2x + 2y + 2z$$

(b) *Calculate the curl of* $\vec{F}$, $\operatorname{curl}(\vec{F}) = \vec{\nabla}\times\vec{F}$.

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - z^2 & y^2 - x^2 & z^2 - y^2 \end{vmatrix}$$
$$= \vec{\imath}(-2y) - \vec{\jmath}(2z) + \vec{k}(-2x)$$
$$= -2y\vec{\imath} - 2z\vec{\jmath} - 2x\vec{k}.$$

6. (10 points) *Let*

$$f(x,y) = xy$$

and let $\mathcal{R}$ *be the region in the plane (see Figure 1) bounded by the lines*

$$y = 0$$
$$y = x$$
$$y = 2 - x.$$

Set up and evaluate the double integral $\iint_{\mathcal{R}} f(x,y)\, dA$.

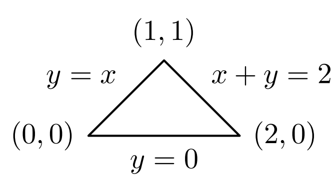

Figure 1: Problem 6

Version 1: If we slice vertically, we need to divide $\mathcal{R}$ into two regions, since the top edge is defined by two different formulas:

$$\mathcal{R}_1 : \begin{cases} 0 & \leq y \leq & x \\ 0 & \leq x \leq & 1 \end{cases}$$

and

$$\mathcal{R}_2 : \begin{cases} 0 & \leq y \leq & 2-x \\ 1 & \leq x \leq & 2 \end{cases}$$

Then

$$\begin{aligned} \iint_{\mathcal{R}_1} xy\,dA &= \int_0^1 \int_0^x xy\,dy\,dx \\ &= \int_0^1 \frac{xy^2}{2}\Big|_{y=0}^{x}\,dx \\ &= \int_0^1 \frac{x^3}{2}\,dx \\ &= \frac{x^4}{8} \\ &= \frac{1}{8}; \end{aligned}$$

$$\begin{aligned} \iint_{\mathcal{R}_2} xy\,dA &= \int_0^2 \int_0^{2-x} xy\,dy\,dx \\ &= \int_1^2 \frac{xy^2}{2}\Big|_{y=0}^{2-x}\,dx \\ &= \int_1^2 \frac{x}{2}(2-x)^2\,dx \\ &= \int_1^2 \left(2x - 2x^{2+\frac{x^3}{2}}\right)\,dx \\ &= \left(x^2 - \frac{2x^3}{3} + \frac{x^4}{8}\right)_1^2 \\ &= \frac{2}{3} - \frac{11}{24} \\ &= \frac{5}{24}; \end{aligned}$$

and so

$$\iint_{\mathcal{R}} xy\,dA = \iint_{\mathcal{R}_1} xy\,dA + \iint_{\mathcal{R}_2} xy\,dA = \frac{1}{8} + \frac{5}{24} = \frac{1}{3}.$$

Version 2: Slicing horizontally, we get a single specification of $\mathcal{R}$:

$$\begin{cases} y & \leq x \leq & 2-y \\ 0 & \leq y \leq & 1 \end{cases}$$

Then

$$\iint_{\mathcal{R}} xy\,dA = \int_0^1 \int_y^{2-y} xy\,dx\,dy$$
$$= \int_0^1 \frac{x^2y}{2}\Big|_{x=y}^{2-y} dy$$
$$= \int_0^1 \left(\frac{(2-y)^2}{2}\cdot y - \frac{y^3}{2}\right) dy$$
$$= \int_0^1 (2y - y^2)\,dy$$
$$= \left(y^2 - \frac{2y^3}{3}\right)_{y=0}^{1}$$
$$= 1 - \frac{2}{3}$$
$$= \frac{1}{3}.$$

7. (13 points) *Consider the vector field*

$$\overrightarrow{F}(x,y) = (3x^2 - 2y + 2)\overrightarrow{\imath} + (2y - 2x + 1)\overrightarrow{\jmath}.$$

(a) *Either show that* $\overrightarrow{F}$ *is* not *conservative, or find a potential function for* $\overrightarrow{F}$*.*

Since
$$\frac{\partial Q}{\partial x} = -2 = \frac{\partial P}{\partial y},$$
the vector field *is* conservative; we need to find a potential.

A function satisfying the first condition

$$f_x = 3x^2 - 2y + 2$$

has the form

$$f(x,y) = x^3 - 2xy + 2x + C(y);$$

for such a function, the second condition reads

$$2y - 2x + 1 = f_y = -2x + C'(y)$$

so

$$2y + 1 = C'(y)$$
$$C(y) = y^2 + y + C$$

and a function satisfying both conditions has the form

$$f(x,y) = x^3 - 2xy + 2x + y^2 + y + C.$$

(b) *Evaluate the line integral*

$$\int_{\mathcal{C}} \overrightarrow{F} \cdot \overrightarrow{T}\, ds = \int_{\mathcal{C}} \overrightarrow{F} \cdot d\vec{r}$$

where $\mathcal{C}$ is the path going in a straight line from $(-1,0)$ to $(1,0)$.

Version 1: Using the Fundamental Theorem for Line Integrals together with our preceding calculation,

$$\begin{aligned} f(\overrightarrow{r}(-1)) &= f(-1,0) \\ &= -1-2 = -3 \end{aligned}$$

and

$$\begin{aligned} f(\overrightarrow{r}(1)) &= f(-1,0) \\ &= 1+2 = 3 \end{aligned}$$

so by FTLI,

$$\begin{aligned} \int_{\mathcal{C}} \overrightarrow{F} \cdot dr &= f(1,0) - f(-1,0) \\ &= 3 - (-3) \\ &= 6. \end{aligned}$$

Version 2: This integral can also be calculated directly. A natural parametrization of $\mathcal{C}$ is

$$\begin{cases} x = t \\ y = 0 \end{cases} \qquad -1 \le t \le 1.$$

Then

$$d\vec{r} = dt\,\overrightarrow{\imath}$$

and the vector field along the curve is

$$\overrightarrow{F}(t,0) = (3t^2+2)\overrightarrow{\imath} + (1-2t)\overrightarrow{\jmath}$$

so

$$\begin{aligned} \int_{\mathcal{C}} \overrightarrow{F} \cdot d\vec{r} &= \int_{-1}^{1} (3t^2+2)\, dt \\ &= (t^3+2t)_{-1}^{1} \\ &= (1+2) - (-1-2) \\ &= 6. \end{aligned}$$

8. (12 points) *Consider the planar vector field*

$$\overrightarrow{F}(x,y) = (x-y)\overrightarrow{\imath} + (2x+y)\overrightarrow{\jmath}$$

and the closed, directed curve $\mathcal{C}$ consisting of the arc of the parabola $y = x^2$ from $(-2,4)$ to $(2,4)$ followed by the straight line segment from $(2,4)$ to $(-2,4)$ (See Figure 2.)

(a) *Set up an* iterated *(double) integral equal to the circulation integral*

$$\oint_{\mathcal{C}} \overrightarrow{F} \cdot \overrightarrow{T}\, ds = \oint_{\mathcal{C}} \overrightarrow{F} \cdot d\vec{r}.$$

By Green's Theorem,

$$\begin{aligned}
\oint_{\mathcal{C}} \overrightarrow{F} \cdot \overrightarrow{T}\, ds &= \iint_{\mathcal{R}} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dV \\
&= \int_{-2}^{2} \int_{x^2}^{4} (\frac{\partial 2x + y}{\partial x} - \frac{\partial x - y}{\partial y})\, dy\, dx \\
&= \int_{-2}^{2} \int_{x^2}^{4} (2 - (-1))\, dy\, dx \\
&= \int_{-2}^{2} \int_{x^2}^{4} 3\, dy\, dx
\end{aligned}$$

(b) *Evaluate* $\oint_{\mathcal{C}} \overrightarrow{F} \cdot \overrightarrow{T}\, ds$:
Version 1: The double integral can be evaluated as

$$\begin{aligned}
\oint_{\mathcal{C}} \overrightarrow{F} \cdot \overrightarrow{T}\, ds &= \int_{-2}^{2} \int_{x^2}^{4} 3\, dy\, dx \\
&= \int_{-2}^{2} (12 - 3x^2)\, dx \\
&= (12x - x^3)\Big|_{-2}^{2} \\
&= (24 - 8) - (-24 + 8) \\
&= 32.
\end{aligned}$$

Version 2: This can also be done directly. We need to break $\mathcal{C}$ into two parts:
$\boldsymbol{\mathcal{C}_1}$**:** The parabola from $(-2, 4)$ to $(2, 4)$ has a natural parametrization

$$\begin{cases} x &= t \\ y &= t^2 \end{cases} \qquad -2 \leq t \leq 2.$$

Thus

$$d\vec{r} = (\overrightarrow{\imath} + 2t\overrightarrow{\jmath})\, dt$$

and the vector field along the curve is

$$\overrightarrow{F}(t, t^2) = (t - t^2)\overrightarrow{\imath} + (2t + t^2)\overrightarrow{\jmath}$$

so

$$\begin{aligned}
\overrightarrow{F} \cdot d\vec{r} &= \{(t - t^2) \cdot 1 + (2t + t^2) \cdot (2t)\}\, dt \\
&= \{t - t^2 + 4t^2 + 2t^3\}\, dt \\
&= (t + 3t^2 + 2t^3)\, dt.
\end{aligned}$$

Thus

$$\int_{\mathcal{C}_1} \overrightarrow{F} \cdot d\vec{r} = \int_{-2}^{2} (t + 3t^2 + 2t^3)\, dt$$
$$= \left(\frac{t^2}{2} + t^3 + \frac{t^4}{2} \right)_{-2}^{2}$$
$$= 16.$$

$\mathcal{C}_2$: The straight line from $(2,4)$ to $(-2,4)$ has parametrization

$$\begin{cases} x &= 2 - t \\ y &= 4 \end{cases} \qquad 0 \le t \le 4.$$

Thus

$$d\vec{r} = -\overrightarrow{\imath}\, dt$$

and the vector field along the curve is

$$\begin{aligned} \overrightarrow{F}(2-t,4) &= (2-t-4)\overrightarrow{\imath} + (4-2t+4)\overrightarrow{\jmath} \\ &= (-t-2)\overrightarrow{\imath} + (8-2t)\overrightarrow{\jmath} \end{aligned}$$

so

$$\begin{aligned} \overrightarrow{F} \cdot d\vec{r} &= \{(-t-2)\cdot(-1) + (8-2t)\cdot(0)\}\, dt \\ &= \{t+2\}\, dt. \end{aligned}$$

Thus

$$\int_{\mathcal{C}_2} \overrightarrow{F} \cdot d\vec{r} = \int_{0}^{4} (t+2)\, dt$$
$$= \left(\frac{t^2}{2} + 2t \right)_{0}^{4}$$
$$= 16.$$

Thus

$$\oint_{\mathcal{C}} \overrightarrow{F} \cdot d\vec{r} = \int_{\mathcal{C}_1} \overrightarrow{F} \cdot d\vec{r} + \int_{\mathcal{C}_2} \overrightarrow{F} \cdot d\vec{r} = 16 + 16 = 32.$$

9. (10 points) Find the (outward) flux

$$\iint_{\mathcal{S}} \overrightarrow{F} \cdot \overrightarrow{n}\, d\mathcal{S} = \iint_{\mathcal{S}} \overrightarrow{F} \cdot d\overrightarrow{\mathcal{S}}$$

of the vector field

$$\overrightarrow{F}(x,y,z) = x\,\overrightarrow{\imath} + y\,\overrightarrow{\jmath} + z\,\overrightarrow{k}$$

over the boundary $\mathcal{S} = \partial\mathcal{D}$ of the solid $\mathcal{D}$ bounded below by the cone

$$z = \sqrt{x^2 + y^2}$$

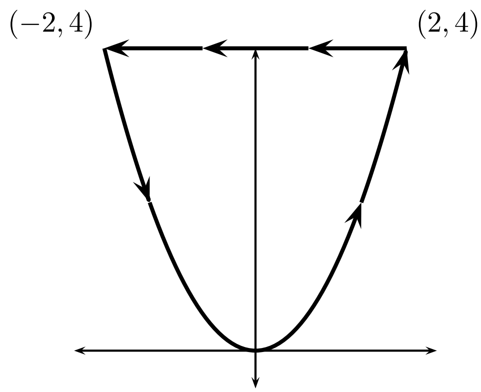

Figure 2: Problem 8

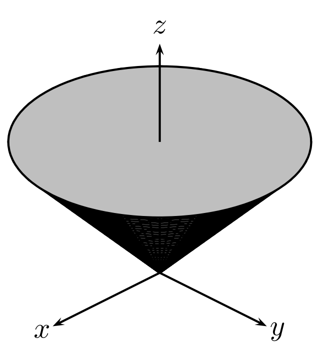

Figure 3: Problem 9

and above by the plane

$$z = 2.$$

(See Figure 3.) (*Hint:* Note that the boundary consists of both the cone and the disc. You may evaluate the flux directly, but consider using the Divergence Theorem.)

Via Divergence Theorem: The divergence of the vector field is

$$\operatorname{div} \overrightarrow{F} = \overrightarrow{\nabla} \cdot \overrightarrow{F} = \frac{dx}{dx} + \frac{dy}{dY} + \frac{dz}{dz} = 3.$$

Thus, by the Divergence Theorem, the outward flux of $\overrightarrow{F}$ over the boundary of $\mathcal{D}$ equals the integral over $\mathcal{D}$ of the divergence,

$$\begin{aligned}
\iiint_{\mathcal{D}} \overrightarrow{\nabla} \cdot \overrightarrow{F}\, dV &= \iiint_{\mathcal{D}} 3\, dV \\
&= 3 \int_0^{2\pi} \int_0^2 \int_r^2 r\, dz\, dr\, d\theta \\
&= 3 \int_0^{2\pi} \int_0^2 (2r - r^2)\, dr\, d\theta \\
&= 3 \int_0^{2\pi} \left(r^2 - \frac{r^3}{3} \right)_0^2 d\theta \\
&= 3 \left(4 - \frac{8}{3} \right) \\
&= 8\pi.
\end{aligned}$$

Direct Calculation: TOP: Along the top boundary of the solid, the vector field has the form

$$\overrightarrow{F}(x, y, 2) = x\,\overrightarrow{\imath} + y\,\overrightarrow{\jmath} + 2\,\overrightarrow{k}$$

and using either the fact that this is a horizontal disc or that it is the graph of a constant function,

$$d\vec{S} = \overrightarrow{k}\, dx\, dy$$

so

$$\begin{aligned}
\iint_{TOP} \overrightarrow{F} \cdot d\vec{S} &= \iint_{x^2+y^2 \le 4} 2\, dx\, dy \\
&= 4(2\pi) \\
&= 8\pi.
\end{aligned}$$

BOTTOM: The bottom boundary of the solid is the graph of the function

$$g(x, y) = \sqrt{x^2 + y^2}.$$

Along this surface, the vector field has the form

$$\overrightarrow{F}\left(x, y, \sqrt{x^2 + y^2}\right) = x\,\overrightarrow{\imath} + y\,\overrightarrow{\jmath} + (\sqrt{x^2 + y^2})\,\overrightarrow{k}.$$

The standard formula for $d\vec{S}$ in the case of a graph is for the *upward* orientation, but since this is the bottom boundary, the outward orientation is *downward*, so we need to reverse the sign:

$$\begin{aligned} d\vec{S} &= -(-g_x \overrightarrow{\imath} - g_y \overrightarrow{\jmath} + \overrightarrow{k}\,)\,dx\,dy \\ &= \left(\frac{x \overrightarrow{\imath} + y \overrightarrow{\jmath}}{\sqrt{x^2+y^2}} - \overrightarrow{k} \right) dx\,dy \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{F} \cdot d\vec{S} &= \left(\frac{x^2+y^2}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2} \right) dx\,dy \\ &= 0 \end{aligned}$$

and hence the flux integral over this part of the boundary cancels out:

$$\iint_{BOTTOM} \overrightarrow{F} \cdot d\vec{S} = 0.$$

Thus,

$$\begin{aligned} \iint_{\partial \mathcal{D}} \overrightarrow{F} \cdot d\vec{S} &= \iint_{TOP} \overrightarrow{F} \cdot d\vec{S} + \iint_{BOTTOM} \overrightarrow{F} \cdot d\vec{S} \\ &= 8\pi + 0 \\ &= 8\pi. \end{aligned}$$