Math 42 | Tufts University, Department of Mathematics, Exam 2 | November 5, 2012, 12:00 noon to 1:20 pm

Instructions: No calculators, notes or books are allowed. You must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (15 points) Consider the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)} dy\, dx$.
   (a) Rewrite this integral as an equivalent iterated integral in polar coordinates.
   (b) Evaluate this integral

2. (10 points) Evaluate the integral $\int_0^1 \int_0^{\frac{\pi}{2}} x\cos(xy)\, dx\, dy$.

3. (15 points) Find the absolute maximum and the absolute minimum of the function

$$f(x,y) = x^2 + y^2 - 2x + 2y + 5$$

   on the closed disk of radius 2 centered at the origin

$$D = \{(x,y)|x^2 + y^2 \leq 4\}$$

4. (15 points) Use the Method of Lagrange Multipliers to find the smallest distance between the origin and a point on the (hyperbolic) cylinder $x^2 + 4x - z^2 + 3 = 0$. Hint: consider the square of the distance instead. *No credit will be given for a solution that does not use Lagrange Multipliers!*

5. (20 points) Consider the icecream cone bounded above by the sphere $x^2 + y^2 + z^2 = 1$ below by the cone $z = \sqrt{3x^2 + 3y^2}$.
   (a) Write an iterated integral in cylindrical coordinates that gives the volume of icecream. DO NOT EVALUATE.
   (b) Write an iterated integral in spherical coordinates that gives the volume of icecream. DO NOT EVALUATE.

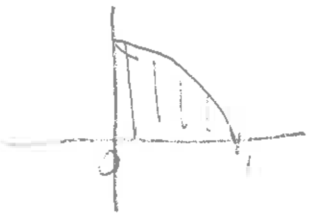
6. (15 points) The wedge of the parabolic cylinder $y = x^2$ cut by the planes $z = 0$ and $z = 3 - y$ has density at each point $f(x,y,z) = x$. Set up an iterated triple integral, including limits of integration for the mass of this wedge. DO NOT EVALUATE.

7. (10 points) Evaluate the integral

$$\int\int\int_D (x^2 + y^2 + z^2)^{\frac{3}{2}} dV$$

   where D is the region in the first octant between the two spheres of radius 1 and 2 centered at the origin.

(1) $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^1 r e^{-r^2}\,dr\,d\theta = \frac{\pi}{2}\left[-\frac{e^{-r^2}}{2}\right]_0^1 =$

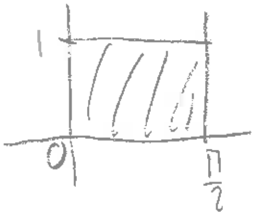


$y = \sqrt{1-x^2}$

$y^2 + x^2 = 1$

$= \frac{\pi}{2}\left(\frac{1}{2} - \frac{e^{-1}}{2}\right) = \frac{\pi}{4}\left(1 - \frac{1}{e}\right)$

(2)



$\int_0^1 \int_0^{\frac{\pi}{2}} x\cos(xy)\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^1 x\cos(xy)\,dy\,dx =$

$= \int_0^{\frac{\pi}{2}} \sin(xy)\Big|_{y=0}^{y=1} dx = \int_0^{\frac{\pi}{2}} (\sin x - \sin 0)\,dx =$

$= -\cos x \Big|_0^{\frac{\pi}{2}} = -0 + 1 = 1$

(3) $f(x,y) = x^2 + y^2 - 2x + 2y + 5$

$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x - 2 = 0 \quad x = 1 \\ \frac{\partial f}{\partial y} &= 2y + 2 = 0 \quad y = -1 \end{aligned}\right\}$

Boundary $(2\cos t, 2\sin t)$

$f(t) = 4 - 2\cos t + 2\sin t + 5 = 6 + 2\sin t - 2\cos t$

$f'(t) = 2\cos t + 2\sin t = 0$ if $\sin t = -\cos t$ $\quad t = \frac{3\pi}{4}, -\frac{\pi}{4}$

Possible points $(1,-1)$, $(\sqrt{2}, -\sqrt{2})$ $\quad (-\sqrt{2}, \sqrt{2})$

$f(1,-1) = 2 - 2 - 2 + 5 = 3$ $\qquad f(\sqrt{2}, -\sqrt{2}) = 4 - 2\sqrt{2} - 2\sqrt{2} + 5 = 9 - 4\sqrt{2}$

$f(-\sqrt{2}, \sqrt{2}) = 4 + 2\sqrt{2} + 2\sqrt{2} + 5 = 9 + 4\sqrt{2} > 3$

$9 - 4\sqrt{2} \overset{?}{<} 3 \iff 6 < 4\sqrt{2} \iff 36 < 32$ false

So absolute max of $9 + 4\sqrt{2}$ at $(-\sqrt{2}, \sqrt{2})$

$\qquad$ " $\qquad$ min of 3 at $(1,-1)$

(4) Distance^2 to origin $= f(x,y,z) = x^2 + y^2 + z^2$ $\qquad \vec{\nabla} f \parallel \nabla g$

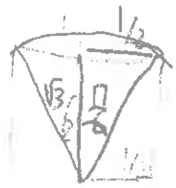
Constraint $g(x,y,z) = x^2 + 4x - z^2 + 3 = 0$

$\langle 2x, 2y, 2z \rangle \parallel \langle 2x+4, 0, -2z \rangle \Rightarrow y = 0 \quad \frac{2x+4}{2x} = \frac{-2z}{2z}$

$\left\{\begin{aligned} &z = 0 \Rightarrow x^2 + 4x + 3 = 0 \quad (x+2)^2 = 1 \Rightarrow x = -1 \text{ or } x = -3 \\ &\text{or } \frac{2x+4}{2x} = -1 \Rightarrow 2x + 4 = -2x \Rightarrow x = -1 \quad 1 - 4 - z^2 + 3 = 0 \Rightarrow z = 0 \end{aligned}\right.$

$(-1, 0, 0)$ or $(-3, 0, 0)$ $\qquad$ minimum distance is 1 at $\boxed{(-1,0,0)}$

⑤ $x^2+y^2+z^2=1$ $\quad z=\sqrt{3x^2+3y^2}=\sqrt{3}r$, sphere ρ

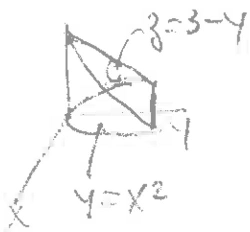


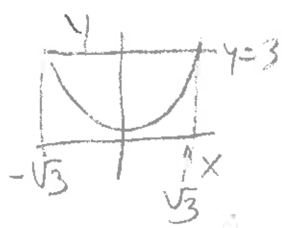
Intersection $x^2+y^2+3x^2+3y^2=1$ $\quad 4(x^2+y^2)=1$

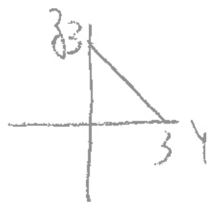
$4r^2=1 \quad r=\frac{1}{2} \quad z=\sqrt{3\left(\frac{1}{2}\right)^2}=\frac{\sqrt{3}}{2}$

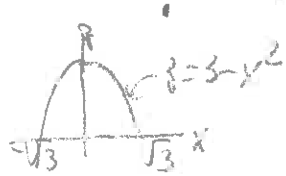
$$\int_0^{2\pi}\int_0^{1/2}\int_{\sqrt{3}r}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

⑥ $y=x^2$, $z=0$, $z=3-y$









$z=3-y=3-x^2$
$y=x^2$

$$\int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3}\int_0^{3-y} x\,dz\,dy\,dx =$$
$$=\int_0^3\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{3-y} x\,dz\,dx\,dy$$
$$=\int_0^3\int_0^{3-y}\int_{-\sqrt{y}}^{\sqrt{y}} x\,dx\,dz\,dy =$$
$$=\int_0^3\int_0^{3-z}\int_{-\sqrt{y}}^{\sqrt{y}} x\,dx\,dy\,dz =$$
$$=\int_{-\sqrt{3}}^{\sqrt{3}}\int_0^{3-x^2}\int_{x^2}^{3-z} x\,dy\,dz\,dx =$$
$$=\int_0^3\int_{-\sqrt{3-z}}^{\sqrt{3-z}}\int_{x^2}^{3-z} x\,dy\,dx\,dz$$

⑦ $\iiint_D (x^2+y^2+z^2)^{3/2}\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^2 \rho^3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta =$

$=\frac{\pi}{2}\left[-\cos\phi\right]_0^{\frac{\pi}{2}} \left.\frac{\rho^6}{6}\right]_1^2 = \frac{\pi}{2}\,\frac{2^6-1}{6} = \frac{\pi}{24}(2^6-1) = \frac{63\pi}{24} = \frac{21\pi}{4}$