Math 13	Tufts University Department of Mathematics Final Exam	December 16, 2011 8:30 to 10:30 am

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in $\mathbb{R}^3$. If $\mathbf{a}$ is orthogonal to $\mathbf{b}$ and $\mathbf{b}$ is orthogonal to $\mathbf{c}$, then $\mathbf{a}$ is orthogonal to $\mathbf{c}$.
   Solution: False: Take $\mathbf{a} = \langle 1, 0, 0\rangle$, $\mathbf{b} = \langle 0, 0, 1\rangle$, and $\mathbf{c} = \langle 1, 1, 0\rangle$.

   (b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|$.
   Solution: True: from definition of the cross product, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\theta = \pi/2$, and $\sin\theta = 1$.

   (c) The volume of the region given in spherical coordinates by

   $$D = \left\{(\rho, \varphi, \theta) \mid 0 \le \rho \le 2, \frac{\pi}{4} \le \varphi \le \frac{\pi}{2}, 0 \le \theta \le \pi\right\}$$

   is $\displaystyle\int_0^{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 1\, d\rho\, d\varphi\, d\theta$.
   Solution: False: The volume is given by $\iiint_D 1 dV$, but when this is converted into spherical coordinates, we need to add $\rho^2 \sin\varphi$ to the integrand:

   $$V(D) = \int_0^{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

   (d) Let $\mathbf{F} = \langle f, g, h\rangle$ be a vector field where f, g, and h have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.
   Solution: True: from calculation shown in class

   (e) If $\mathcal{C}$ is a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region, R, in the plane, then the area of R is given by -$\oint_{\mathcal{C}} y dx$.
   Solution: True, using Green's Theorem area formula.

2. (10 points) Let $\mathbf{v} = \langle 2, -2, 1\rangle$.

   (a) Find all vectors of the form $\langle a, 1, b\rangle$ orthogonal to $\mathbf{v}$.
   Solution: For these vectors to be orthogonal, we need $\langle a, 1, b\rangle \cdot \langle 2, -2, 1\rangle = 2a - 2 + b = 0$, which gives $b = 2 - 2a$. Thus, the vectors

   $$\langle a, 1, 2 - 2a\rangle$$

   are orthogonal to $\mathbf{v}$ for all values of a.

(b) Let $\mathbf{u} = \langle 1, 2, 0 \rangle$. Calculate $\texttt{scal}_{\mathbf{v}}\mathbf{u}$ and $\texttt{proj}_{\mathbf{v}}\mathbf{u}$.
Solution: From the definitions,

$$\texttt{scal}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{\langle 1, 2, 0 \rangle \cdot \langle 2, -2, 1 \rangle}{\sqrt{2^2 + (-2)^2 + 1^2}} = -\frac{2}{3}$$

$$\texttt{proj}_{\mathbf{v}}\mathbf{u} = (\texttt{scal}_{\mathbf{v}}\mathbf{u}) \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{2}{3}\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle = \left\langle -\frac{4}{9}, \frac{4}{9}, \frac{2}{9} \right\rangle$$

3. (10 points) Let $f(x, y) = x\sqrt{y}$.

(a) Find the linear approximation to $f(x, y)$ at the point $(3, 4)$.
Solution: The linear approximation to $f(x, y)$ at the point (x_0, y_0) is given by

$$\ell(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Computing, we get

$$\frac{\partial f}{\partial x}(x, y) = \sqrt{y}, \text{ and } \frac{\partial f}{\partial y}(x, y) = \frac{x}{2\sqrt{y}}$$

$$\text{and, so, } \frac{\partial f}{\partial x}(3, 4) = 2 \quad , \text{ and } \frac{\partial f}{\partial y}(x, y) = \frac{3}{4}.$$

Plugging these in gives

$$\ell(x, y) = 6 + 2(x - 3) + \frac{3}{4}(y - 4).$$

(b) Use the linear approximation to estimate the value of $3.01\sqrt{3.96}$.
Solution: From the formula,

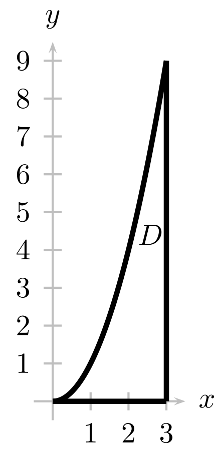
$$\ell(3.01, 3.96) = 6 + 2(3.01 - 3) + \frac{3}{4}(3.96 - 4) = 6 + 0.02 - 0.03 = 5.99.$$

4. (10 points) Consider the integral

$$\int_0^9 \int_{\sqrt{y}}^3 3e^{x^3}\, dx\, dy.$$

(a) Sketch the region of integration.
Solution: From the integral, we know that

$$D = \{(x, y) \mid 0 \leq y \leq 9, \sqrt{y} \leq x \leq 3\}.$$



(b) Evaluate the integral.
Solution: Since we can't evaluate the integral as written, try to switch orders. In this case, from the sketch in Part (a), we see that

$$D = \left\{(x, y) \mid 0 \le x \le 3, 0 \le y \le x^2\right\},$$

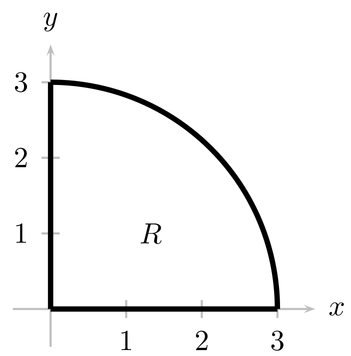
giving

$$\begin{aligned}\int_0^9 \int_{\sqrt{y}}^3 3e^{x^3}\,dx\,dy &= \int_0^3 \int_0^{x^2} 3e^{x^3}\,dy\,dx \\ &= \int_0^3 3x^2 e^{x^3}\,dx = e^{x^3}\Big|_0^3 = e^{27} - 1.\end{aligned}$$

5. (10 points) Suppose

$$\iint_R f(x, y)\,dA = \int_0^{\pi/2} \int_0^3 r^2\,dr\,d\theta.$$

(a) Sketch the region of integration, R, in the xy plane.
Solution: From the iterated integral, $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 3$, meaning that R is the quarter-circle of radius 3 in the first quadrant:



(b) Convert the integral to Cartesian coordinates.
Solution: We can rewrite R in Cartesian coordinates as

$$R = \left\{(x, y) \mid 0 \le x \le 3, 0 \le y \le \sqrt{9 - x^2}\right\}.$$

Then, remembering that $\iint_R f(x, y)dA = \int_0^{\pi/2}\int_0^3 f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$, we recognize $r^2 = r \cdot r = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \cdot r = \sqrt{x^2 + y^2}r$ and have

$$\iint_R f(x, y)\,dA = \int_0^{\pi/2}\int_0^3 r^2\,dr\,d\theta = \int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2}\,dy\,dx.$$

(c) Evaluate the integral. (You may use either polar or Cartesian coordinates.)
Solution: The Cartesian coordinates integral is messy, so consider the polar coordinates version:

$$\int_0^{\pi/2}\int_0^3 r^2\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{3}r^3\Big|_{r=0}^{r=3}\,d\theta = \int_0^{\pi/2} 9\,d\theta = \frac{9\pi}{2}.$$

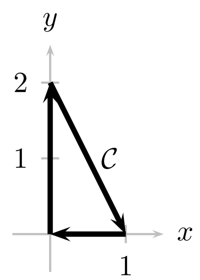
6. (10 points) Let $\mathcal{C}$ be the curve in the plane consisting of the line segments from $(0, 0)$ to $(0, 2)$, from $(0, 2)$ to $(1, 0)$, and from $(1, 0)$ to $(0, 0)$. Use Green's Theorem to evaluate

$$\int_{\mathcal{C}} x^3\,dx + 2xy\,dy.$$

No credit will be given for solutions that do not use Green's Theorem.
Solution: From Green's Theorem, we have

$$\int_{\mathcal{C}} x^3\,dx + 2xy\,dy = \iint_D \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^3)\,dA,$$

where D is the region bounded by the positively oriented curve, $\mathcal{C}$. Drawing out $\mathcal{C}$, we have



Notice that $\mathcal{C}$ is oriented **clockwise** instead of **counter-clockwise**. This means that the correct calculation is

$$\int_{\mathcal{C}} x^3\,dx + 2xy\,dy = -\int_{-\mathcal{C}} x^3\,dx + 2xy\,dy = -\iint_D \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^3)\,dA,$$

since $-\mathcal{C}$ is the positively-oriented curve that bounds D. Noting that D can be written as

$$D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2 - 2x\},$$

we have

$$\begin{aligned}
\int_{\mathcal{C}} x^3\,dx + 2xy\,dy &= -\iint_D \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^3)\,dA \\
&= -\int_0^1 \int_0^{2-2x} 2y\,dy\,dx = -\int_0^1 y^2\Big|_{y=0}^{y=2-2x}\,dx \\
&= -\int_0^1 4 - 8x + 4x^2\,dx = -\left(4x - 4x^2 + \frac{4}{3}x^3\right)\Bigg|_{x=0}^{x=1} \\
&= -\frac{4}{3}.
\end{aligned}$$

7. (15 points) Consider eating an ice cream cone with no ice cream (yes, it's sad). The surface of the cone, $\mathcal{S}$, is represented by the graph,

$$z = \sqrt{x^2 + y^2} \quad \text{for} \quad 0 \le z \le 2.$$

(a) Consider covering the cone in sprinkles. What is the total surface area of the cone?
Solution: Consider parameterizing the surface as

$$\mathcal{S} = \left\{(x, y, z) \mid z = \sqrt{x^2 + y^2}, (x, y) \text{ in } D\right\},$$

where D is the circle of radius 2 in the xy-plane (so that $0 \le z \le 2$). Then, using the formula for when $z = z(x, y)$, we have

$$\begin{aligned}
\iint_{\mathcal{S}} 1\,dS &= \iint_D \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right)^{\frac{1}{2}} dA \\
&= \iint_D \left(\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1\right)^{\frac{1}{2}} dA = \iint_D \sqrt{2}dA
\end{aligned}$$

Parameterizing D in polar coordinates, $0 \le \theta \le 2\pi$, $0 \le r \le 2$, we get

$$\begin{aligned}
\iint_{\mathcal{S}} 1\,dS &= \int_0^{2\pi} \int_0^2 \sqrt{2}r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{\sqrt{2}}{2}r^2\Bigg|_{r=0}^{r=2} d\theta = \int_0^{2\pi} 2\sqrt{2}\,d\theta \\
&= 4\sqrt{2}\pi.
\end{aligned}$$

(b) Suppose the density of sprinkles that you put on the cone is given by the function, $f(x, y, z) = z^2 + 1$. Compute the total mass of sprinkles on the ice cream cone by evaluating the surface integral,

$$\iint_{\mathcal{S}} f(x, y, z)\,dS.$$

Solution: Using the parameterization above, we have

$$\begin{aligned}
\iint_{\mathcal{S}} f(x,y,z)\,dS &= \iint_{D} \left(\left(\sqrt{x^2+y^2}\right)^2 + 1 \right) \sqrt{2} dA \\
&= \int_0^{2\pi} \int_0^2 \left(r^2+1\right) \sqrt{2} r\,dr\,d\theta = \sqrt{2} \int_0^{2\pi} \int_0^2 r^3 + r\,dr\,d\theta \\
&= \sqrt{2} \int_0^{2\pi} \frac{r^4}{4} + \frac{r^2}{2} \Big|_{r=0}^{r=2} d\theta = \sqrt{2} \int_0^{2\pi} 6\,d\theta \\
&= 12\sqrt{2}\pi.
\end{aligned}$$

8. (15 points) Let $\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z \right\rangle$.

   (a) Compute $\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS$ where $\mathcal{S}$ is the upper half of the unit sphere, $x^2+y^2+z^2=1$, oriented upwards.
   Solution: First, we compute $\nabla \times \mathbf{F}$:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y} z - \frac{\partial}{\partial z} \frac{x}{x^2+y^2} \right) \mathbf{i} - \left(\frac{\partial}{\partial z} \frac{-y}{x^2+y^2} - \frac{\partial}{\partial x} \right) \mathbf{j} + \left(\frac{\partial}{\partial x} \frac{x}{x^2+y^2} - \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} \right) \mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + \left(\frac{x^2+y^2-2x^2}{\left(x^2+y^2\right)^2} + \frac{x^2+y^2-2y^2}{\left(x^2+y^2\right)^2} \right) \mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.
\end{aligned}$$

   Then, $\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \iint_{\mathcal{S}} 0\,dS = 0$.

   (b) Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the unit circle in the xy-plane, oriented counter-clockwise.
   Solution: Parameterize $\mathcal{C}$ by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 0\mathbf{k}$, for $0 \le t \le 2\pi$. Then, from the definition of the line integral,

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left\langle \frac{-\sin(t)}{\cos^2 t + \sin^2 t}, \frac{\cos(t)}{\cos^2 t + \sin^2 t}, 0 \right\rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle\,dt \\
&= \int_0^{2\pi} \frac{\cos^2 t + \sin^2 t}{\cos^2 t + \sin^2 t}\,dt = \int_0^{2\pi} 1\,dt = 2\pi.
\end{aligned}$$

   (c) Compare your two answers. Does Stokes' Theorem apply to these integrals of $\mathbf{F}$?
   Solution: Since the two answers are not the same ($0 \neq 2\pi$), Stokes' Theorem does not apply. This is because $\mathbf{F}$ is not differentiable everywhere on $\mathcal{S}$, as required by the theorem.

9. (10 points) Compute the net outward flux, $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\,dS$, of $\mathbf{F}(x,y,z) = \left\langle y^2, \sin(x), \frac{z^3}{3} \right\rangle$ across the unit sphere, $\mathcal{S}$, given by $x^2+y^2+z^2=1$.
   Solution: The Divergence Theorem tells us that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_{V} \nabla \cdot \mathbf{F} dV,$$

where V is the volume enclosed by the closed surface $\mathcal{S}$; in this case, V is the unit ball, $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$. Computing, we have

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} y^2 + \frac{\partial}{\partial y} \sin(x) + \frac{\partial}{\partial z} \frac{z^3}{3} = z^2.$$

So,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V z^2 dV.$$

Parameterizing the unit ball in the usual way, this gives

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{\pi} \int_0^{2\pi} \int_0^1 \left(\rho^2 \cos^2 \varphi\right) \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^4 \cos^2 \varphi \sin \varphi \, d\rho \, d\theta \, d\varphi = \int_0^{\pi} \int_0^{2\pi} \frac{1}{5} \cos^2 \varphi \sin \varphi \, d\theta \, d\varphi \\
&= \int_0^{\pi} \frac{2\pi}{5} \cos^2 \varphi \sin \varphi \, d\varphi \\
&= -\frac{2\pi}{15} \cos^3 \varphi \Big|_{\varphi=0}^{\varphi=\pi} = \frac{4\pi}{15}.
\end{aligned}$$

End of Exam