**Math 13** | **Tufts University** | **December 16, 2011**

**Department of Mathematics**

**Final Exam** | **8:30 to 10:30 am**

**Instructions:** No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in $\mathbb{R}^3$. If $\mathbf{a}$ is orthogonal to $\mathbf{b}$ and $\mathbf{b}$ is orthogonal to $\mathbf{c}$, then $\mathbf{a}$ is orthogonal to $\mathbf{c}$.

   (b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|$.

   (c) The volume of the region given in spherical coordinates by

   $$D = \left\{(\rho, \varphi, \theta) \mid 0 \leq \rho \leq 2, \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}, 0 \leq \theta \leq \pi\right\}$$

   is $\displaystyle\int_0^{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 1\, d\rho\, d\varphi\, d\theta$.

   (d) Let $\mathbf{F} = \langle f, g, h\rangle$ be a vector field where $f$, $g$, and $h$ have continuous second partial derivatives. Then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

   (e) If $\mathcal{C}$ is a simple closed smooth curve, oriented counterclockwise, that encloses a connected and simply connected region, $R$, in the plane, then the area of $R$ is given by -$\oint_{\mathcal{C}} y dx$.

2. (10 points) Let $\mathbf{v} = \langle 2, -2, 1\rangle$.

   (a) Find all vectors of the form $\langle a, 1, b\rangle$ orthogonal to $\mathbf{v}$.

   (b) Let $\mathbf{u} = \langle 1, 2, 0\rangle$. Calculate $\mathtt{scal}_{\mathbf{v}}\mathbf{u}$ and $\mathtt{proj}_{\mathbf{v}}\mathbf{u}$.

3. (10 points) Let $f(x, y) = x\sqrt{y}$.

   (a) Find the linear approximation to $f(x, y)$ at the point $(3, 4)$.

   (b) Use the linear approximation to estimate the value of $3.01\sqrt{3.96}$.

4. (10 points) Consider the integral

   $$\int_0^9 \int_{\sqrt{y}}^3 3e^{x^3}\, dx\, dy.$$

   (a) Sketch the region of integration.

   (b) Evaluate the integral.

**The exam continues on the back!**

5. (10 points) Suppose

$$\iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_0^3 r^2\,dr\,d\theta.$$

   (a) Sketch the region of integration, $R$, in the $xy$ plane.

   (b) Convert the integral to Cartesian coordinates.

   (c) Evaluate the integral. (You may use either polar or Cartesian coordinates.)

6. (10 points) Let $\mathcal{C}$ be the curve in the plane consisting of the line segments from $(0,0)$ to $(0,2)$, from $(0,2)$ to $(1,0)$, and from $(1,0)$ to $(0,0)$. Evaluate $\int_{\mathcal{C}} x^3\,dx + 2xy\,dy$ using Green's Theorem.

7. (15 points) Consider eating an ice cream cone with no ice cream (yes, it's sad). The surface of the cone, $\mathcal{S}$, is represented by the graph,

$$z = \sqrt{x^2+y^2} \quad \text{for} \quad 0 \le z \le 2.$$

   (a) Consider covering the cone in sprinkles. What is the total surface area of the cone?

   (b) Suppose the density of sprinkles that you put on the cone is given by the function, $f(x,y,z) = z^2+1$. Compute the total mass of sprinkles on the ice cream cone by evaluating the surface integral,

$$\iint_{\mathcal{S}} f(x,y,z)\,dS.$$

8. (15 points) Let $\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z \right\rangle$.

   (a) Compute $\iint_{\mathcal{S}} (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS$ where $S$ is the upper half of the unit sphere, $x^2+y^2+z^2=1$, oriented upwards.

   (b) Compute $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ where $\mathcal{C}$ is the unit circle in the $xy$-plane, oriented counter-clockwise.

   (c) Compare your two answers. Does Stokes' Theorem apply to these integrals of $\mathbf{F}$?

9. (10 points) Compute the net outward flux, $\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,dS$, of $\mathbf{F}(x,y,z) = \left\langle y^2, \sin(x), \frac{z^3}{3} \right\rangle$ across the unit sphere, $\mathcal{S}$, given by $x^2+y^2+z^2=1$.

**End of Exam**