Tufts University

Math 13 — **Department of Mathematics** — **November 14, 2011**

Exam 1 — **12:00 noon to 1:20 pm**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) $\int_0^1 \int_0^x \sqrt{x+y^2}\,dy\,dx = \int_0^x \int_0^1 \sqrt{x+y^2}\,dx\,dy$
   Solution: False: the outside integral on the right has an x in its limits, which isn't allowed.

   (b) The point $(x, y, z) = (1, \frac{1}{\sqrt{3}}, 1)$ in Cartesian coordinates is the same as $r = \frac{2}{\sqrt{3}}$, $\theta = \frac{\pi}{6}$, $z = 6$ in cylindrical coordinates.
   Solution: False: the z-coordinates of the two points are different.

   (c) The point $(x, y, z) = (\sqrt{2}, \sqrt{2}, 0)$ in Cartesian coordinates is the same as $\rho = 2$, $\varphi = \frac{\pi}{2}$, and $\theta = \frac{\pi}{4}$ in spherical coordinates.
   Solution: True: First note that $\varphi = \frac{\pi}{2}$ means that the point is in the $xy-$plane, so has coordinate $z = 0$. Then compute from the formulas: $x = \rho\cos\theta\sin\varphi = \sqrt{2}$, $y = \rho\sin\theta\sin\varphi = \sqrt{2}$.

   (d) The vector field $\mathbf{F}(x, y, z) = \langle 0, 0, 0\rangle$ is conservative.
   Solution: True: a potential function for it is $f(x, y, z) = 1$.

   (e) If $\mathbf{F}$ is a conservative field and C is parameterized by $\mathbf{r}(t)$, $a \le t \le b$, then $\int_C \mathbf{F}\cdot d\mathbf{r} = \mathbf{F}(\mathbf{r}(b)) - \mathbf{F}(\mathbf{r}(a))$.
   Solution: False: The integral on the left should give a scalar as an answer, since it is the line integral of a vector field. The values on the right, however, are vectors.

2. (10 points)

   (a) Express the volume of the solid region, E, that sits above the rectangle in the xy-plane with vertices $(1, 1, 0)$, $(4, 1, 0)$, $(1, 2, 0)$, and $(4, 2, 0)$ and below the surface $z = xy$ in terms of a *double* integral.
   Solution: The volume of the region between $z = 0$ and $z = xy$ within the rectangle $[1, 4] \times [1, 2]$ in the $xy-$plane is given by the double integral

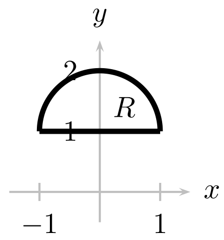
$$\int_1^4 \int_1^2 xydydx.$$

   (b) Evaluate the integral you found in part (a).
   Solution:

$$\int_1^4 \int_1^2 xydydx = \int_1^4 \frac{xy^2}{2}\bigg|_{y=1}^{y=2} dx$$
$$= \int_1^4 \frac{3}{2}xdx = \frac{3}{4}x^2\bigg|_1^4 = \frac{45}{4}.$$

3. (15 points) Let R be the region in the plane inside the circle $x^2 + y^2 = 2y$ and above the line $y = 1$.

   (a) Sketch R.
   Solution: The circle $x^2 + y^2 = 2y$ can be rewritten as $x^2 + (y-1)^2 = 1$, so it is the circle of radius 1 centered at $(0, 1)$. The upper half of this circle is above the line $y = 1$.



   (b) Express R in polar coordinates.
   Solution: Note that we can rewrite the line $y = 1$ as $y = r\sin\theta = 1$, or $r = \frac{1}{\sin\theta}$. We can also rewrite the circle $x^2 + y^2 = 2y$ as $r^2 = 2r\sin\theta$, or $r = 2\sin\theta$. Thus, we can write both the top and bottom boundaries of R by giving r in terms of θ. We now need to find limits on θ. To do this, notice that the limits of the region in θ will occur when $r = \frac{1}{\sin\theta}$ and $r = 2\sin\theta$ are both satisfied: $\frac{1}{\sin\theta} = 2\sin\theta$, which gives $\sin^2\theta = \frac{1}{2}$, or $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. (There are, of course, other values of θ that satisfy $\sin^2\theta = \frac{1}{2}$, but the region, R, only extends into the first and second quadrants.
   Putting this together gives $R = \left\{(r, \theta) \mid \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}, \frac{1}{\sin\theta} \le r \le 2\sin\theta\right\}$.

   (c) Compute $\iint_R f(x, y)dA$ for

$$f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}.$$

   Solution: First, we make the change of variables from (x, y) to (r, θ):

$$\iint_R f(x, y)dA = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\frac{1}{\sin\theta}}^{2\sin\theta} \frac{r\cos\theta}{r} r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\frac{1}{\sin\theta}}^{2\sin\theta} r\cos\theta\,dr\,d\theta.$$

   Integrating with respect to r, we get

$$\iint_R f(x, y)dA = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left.\frac{r^2}{2}\right|_{r=\frac{1}{\sin\theta}}^{r=2\sin\theta} \cos\theta\,d\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(4\sin^2\theta - \frac{1}{\sin^2\theta}\right)\cos\theta\,d\theta.$$

   To evaluate this, notice that we can easily make a u-substitution for $u = \sin\theta$, giving $du = \cos\theta\,d\theta$, and

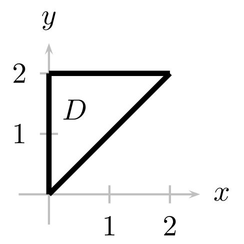
$$\iint_R f(x, y)dA = \frac{1}{2}\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 4u^2 - \frac{1}{u^2}du = 0.$$

4. (10 points)

   (a) Rewrite the iterated integral

$$\int_0^2 \int_x^2 \int_0^{4-x-y} xy\,dz\,dy\,dx$$

as an iterated integral in the order $dz\,dx\,dy$.
Solution: We only need to exchange the outside order in x and y, so we don't need to worry about the limits in z. So, we draw the sketch of the region, D, in the $xy-$plane:



From the picture, we notice that the maximum limits in y are 0 and 2 while, for fixed y, x varies from 0 to y. This gives

$$\int_0^2 \int_x^2 \int_0^{4-x-y} xy\,dz\,dy\,dx = \int_0^2 \int_0^y \int_0^{4-x-y} xy\,dz\,dx\,dy$$

(b) Evaluate *one* of the integrals from part (a).
Solution: Either integral seems reasonable to compute. Using the second one, we get

$$\begin{aligned}\int_0^2 \int_0^y \int_0^{4-x-y} xy\,dz\,dx\,dy &= \int_0^2 \int_0^y xy(4-x-y)dxdy \\ &= \int_0^2 \int_o^y 4xy - x^2y - xy^2dxdy \\ &= \int_0^2 2x^2y - \frac{1}{3}x^3y - \frac{1}{2}x^2y^2\Big|_{x=0}^{x=y} dy \\ &= \int_0^2 2y^3 - \frac{1}{3}y^4 - \frac{1}{2}y^4dy \\ &= \frac{1}{2}y^4 - \frac{1}{15}y^5 - \frac{1}{10}y^5\Big|_0^2 = \frac{1}{2}y^4 - \frac{1}{6}y^5\Big|_0^2 \\ &= 8 - \frac{32}{6} = \frac{8}{3}.\end{aligned}$$

5. (15 points) Let $f(x, y, z) = z$ and let W be the region that is above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 8$.

(a) Setup (but *do not evaluate*) the integral $\iiint_W f(x, y, z)\,dV$ in both

i. Cylindrical Coordinates, and
Solution: In cylindrical coordinates, the cone $z = \sqrt{x^2 + y^2}$ becomes $z = r$, while the sphere becomes $r^2 + z^2 = 8$. These two surfaces intersect when $r^2 + r^2 = 8$, or $r = 2$. This gives limits on the z integral as $r \le z \le \sqrt{8 - r^2}$ (since W is above the cone and below the sphere), on the r integral as $0 \le r \le 2$ (because the maximum extents in the xy-plane correspond to a circle of radius 2), and on the θ integral as $0 \le \theta \le 2\pi$ (since there are no restrictions on θ. Thus,

$$\iiint_W z\,dV = \int_0^{2\pi} \int_0^2 \int_r^{\sqrt{8-r^2}} zrdzdrd\theta.$$

ii. Spherical Coordinates.

Solution: In spherical coordinates, the sphere is given by $\rho^2 = 8$, or $\rho = 2\sqrt{q}$. The cone is given by

$$\begin{aligned} z &= \sqrt{x^2+y^2} \\ \rho\cos\varphi &= \left(\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi\right)^{\frac{1}{2}} = \rho\sin\varphi. \\ \text{So,}\ \cos\varphi &= \sin\varphi \text{ or } \tan\varphi = \frac{\pi}{4}. \end{aligned}$$

Above (inside) the cone corresponds to angles smaller than $\frac{\pi}{4}$, so the limits on φ are $0 \le \varphi \le \frac{\pi}{4}$. The cone gives no limits on ρ, so the limits on ρ are given by $0 \le \rho \le 2\sqrt{2}$. No limits are given on θ, so $0 \le \theta \le 2\pi$. Thus,

$$\begin{aligned} \iiint_W z\,dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} \rho\cos\varphi\rho^2\sin\varphi d\rho d\varphi d\theta \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} \rho^3\cos\varphi\sin\varphi d\rho d\varphi d\theta. \end{aligned}$$

(b) Evaluate $\iiint_W f(x,y,z)\,dV$ using *one* of the integrals in part (a).

Solution: Either integral is easy to evaluate. In cylindrical coordinates:

$$\begin{aligned} \iiint_W z\,dV &= \int_0^{2\pi}\int_0^2\int_r^{\sqrt{8-r^2}} zrdzdrd\theta \\ &= \int_0^{2\pi}\int_0^2 \frac{1}{2}z^2\Big|_{z=r}^{z=\sqrt{8-r^2}} rdrd\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 8r - 2r^3 drd\theta \\ &= \int_0^{2\pi} 4r^2 - \frac{1}{2}r^4\Big|_{r=0}^{r=2} d\theta = \frac{1}{2}\int_0^{2\pi}(16-8)d\theta \\ &= 8\pi. \end{aligned}$$

In spherical coordinates:

$$\begin{aligned} \iiint_W z\,dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} \rho^3\cos\varphi\sin\varphi d\rho d\varphi d\theta. \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{4}\rho^4\Big|_{\rho=0}^{\rho=\sqrt{8}} \cos\varphi\sin\varphi d\varphi d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} 16\cos\varphi\sin\varphi d\varphi d\theta \\ &= \int_0^{2\pi} 8\sin^2\varphi\Big|_{\varphi=0}^{\varphi=\frac{\pi}{4}} d\theta = \int_0^{2\pi} 4d\theta \\ &= 8\pi. \end{aligned}$$

6. (10 points) Evaluate $\iiint_V \frac{y}{(x^2+y^2)^{1/2}}\,dV$, where

$$V = \{(r,\theta,z) \mid 0 \le r \le 6, 0 \le \theta \le \pi, 0 \le z \le 2/r\}.$$

Solution: Using the cylindrical coordinates description of V given to us, we have

$$\begin{aligned}\iiint_V \frac{y}{(x^2+y^2)^{1/2}}\,dV &= \int_0^{\pi}\int_0^{6}\int_0^{2/r} \frac{r\sin\theta}{r} r dz dr d\theta = \int_0^{\pi}\int_0^{6}\int_0^{2/r} r\sin\theta dz dr d\theta \\ &= \int_0^{\pi}\int_0^{6} 2\sin\theta dr d\theta \\ &= \int_0^{\pi} 12\sin\theta d\theta = -12\cos\theta\big|_0^{\pi} \\ &= -12(-1) - (-12)1 = 24.\end{aligned}$$

7. (10 points) Evaluate the line integral

$$\int_C (3x^2 - 2y^2) ds$$

where C is the curve parameterized by $\mathbf{r}(t) = \langle 4t, 3t\rangle$ for $0 \le t \le 2$.
Solution: From the definition,

$$\int_C f(x,y)ds = \int_0^2 f(4t,3t)|\mathbf{r}'(t)|dt.$$

Computing $\mathbf{r}'(t) = \langle 4, 3\rangle$, so $|vecr'(t)| = 5$, we have

$$\begin{aligned}\int_C (3x^2 - 2y^2)ds &= \int_0^2 \left(3(4t)^2 - 2(3t)^2\right) 5dt \\ &= \int_0^2 5(48t^2 - 18t^2)dt = \int_0^2 150t^2 dt \\ &= 50t^3\big|_0^2 = 400.\end{aligned}$$

8. (10 points) Let $f(x,y,z) = \cos(x)\sin(y)e^z$.

   (a) Compute the gradient field, $\mathbf{F} = \nabla f$.
   Solution: By the definition,

$$\begin{aligned}\mathbf{F} = \nabla f &= \langle f_x, f_y, f_z\rangle \\ &= \langle -\sin(x)\sin(y)e^z, \cos(x)\cos(y)e^z, \cos(x)\sin(y)e^z\rangle .\end{aligned}$$

   (b) Compute the integral, $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is the path given by $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), \cos(t)\rangle$, $0 \le t \le 2\pi$.
   Solution: Since $\mathbf{F}$ is defined as the gradient of $f(x,y,z)$, we know that $\mathbf{F}$ is conservative. The integral of any continuous conservative field over a closed path is always zero, so

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0.$$

   (Note that C is closed: $\mathbf{r}(2\pi) = \mathbf{r}(0)$.)

9. (10 points) Let $\mathbf{F}(x,y,z) = \langle 2xy + e^z, x^2 - \sin(y), xe^z + 7\rangle$. Is $\mathbf{F}$ conservative? If so, find a potential function for $\mathbf{F}$.

Solution: We first check that $\mathbf{F}$ is conservative. Writing $f(x,y,z) = 2xy + e^z$, $g(x,y,z) = x^2 - \sin(y)$, and $h(x,y,z) = xe^z + 7$, we have

$$\frac{\partial f}{\partial y} = 2x = \frac{\partial g}{\partial x}$$
$$\frac{\partial f}{\partial z} = e^z = \frac{\partial h}{\partial x}$$
$$\frac{\partial g}{\partial z} = 0 = \frac{\partial h}{\partial y}.$$

These three conditions verify that $\mathbf{F}$ is conservative.
To find a potential function, $\phi(x,y,z)$, start by noticing that $\frac{\partial \phi}{\partial x} = f(x,y,z) = 2xy + e^z$. So

$$\phi(x,y,z) = \int 2xy + e^z dx = x^2y + xe^z + C(y,z).$$

Differentiating this with respect to y gives $\frac{\partial \phi}{\partial y} = x^2 + \frac{\partial C}{\partial y}$. Matching this with $g(x,y,z)$ gives

$$C(y,z) = \int \left(x^2 - \sin(y)\right) - x^2 dy = \cos(y) + K(z),$$

or

$$\phi(x,y,z) = x^2y + xe^z + \cos(y) + K(z).$$

Differentiating this with respect to z gives $\frac{\partial \phi}{\partial z} = xe^z + \frac{dK}{dz}$. Matching this with $h(x,y,z)$ gives

$$K(z) = \int (xe^z + 7) - xe^z dz = 7z + K.$$

Finally, this gives

$$\phi(x,y,z) = x^2y + xe^z + \cos(y) + 7z + K.$$

To verify that ϕ is a potential function for $\mathbf{F}$, compute

$$\nabla \phi = \langle 2xy + e^z, x^2 - \sin(y), xe^z + 7 \rangle.$$

End of Exam