Tufts University

Math 13	**Department of Mathematics**	**October 17, 2011**
	Exam 1	**12:00 noon to 1:20 pm**

Instructions: No calculators, notes or books are allowed. Unless otherwise stated, you must show all work to receive full credit. **Simplify your answers as much as possible.** Please circle your answers and cross out any work you do not want graded. *You are required to sign your exam book. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

1. (10 points) **True or False - No Partial Credit**: On the first page of your blue book, answer the following questions as **True** or **False**.

   (a) $\mathbf{i} \times (\mathbf{j} \cdot \mathbf{k}) = 1$.
   Solution: False: $\mathbf{j} \cdot \mathbf{k}$ is a scalar, so the cross product on the left-hand side doesn't make sense. Even if it did, the cross product of two vectors is a vector, so it could never equal 1.

   (b) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
   Solution: False, $\mathbf{u}$ and $\mathbf{v}$ could be parallel.

   (c) The function $f(x, y) = xy$ has no critical points.
   Solution: False, $(0, 0)$ is a critical point of $f(x, y)$.

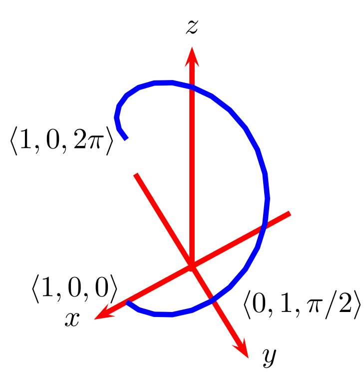
   (d) The gradient of f at a point (a, b) is orthogonal to the level curve of f which contains (a, b).
   Solution: True, as discussed in class.

   (e) The level curves of $z = f(x, y)$ for $f(x, y) = x^2 + y^2$ are parabolas.
   Solution: False, the level curves of $z = x^2 + y^2$ are circles.

2. (8 points) Sketch the curve defined by the vector function

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, 0 \leq t \leq 2\pi.$$

   Solution: Recognize that this is a single rotation of a helix, since $x^2 + y^2 = 1$ and we go through one period of $\cos t$ and $\sin t$.



3. (12 points) An angry bird is moving along a path parameterized by $\mathbf{r}(t)$ with initial position given by

$$\mathbf{r}(0) = \langle -2, 0, 1\rangle$$

and velocity given by

$$\mathbf{v}(t) = \langle 4t\sin(t^2), 4t\cos(t^2), 2t\rangle \text{ for } t \geq 0.$$

(a) Compute the acceleration, $\mathbf{a}(t)$, for the angry bird for any $t \geq 0$.
Solution: To find the acceleration, we differentiate the vector-valued function $\mathbf{v}(t)$:

$$\mathbf{a}(t) = \langle 4(\sin(t^2) + 2t^2\cos(t^2)), 4(\cos(t^2) - 2t^2\sin(t^2)), 2\rangle$$

(b) Compute the trajectory of the angry bird, i.e. its position vector for all $t \geq 0$, $\mathbf{r}(t)$.
Solution: The trajectory of the angry bird is described by the vector-valued function obtained by integrating the velocity, $\mathbf{v}(t)$, and imposing the initial position. First, we integrate, keeping the constant of integration:

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \langle -2\cos(t^2), 2\sin(t^2), t^2\rangle + \mathbf{C}.$$

Then, evaluating at $t = 0$, we get

$$\mathbf{r}(0) = \langle -2, 0, 1\rangle = \langle -2, 0, 0\rangle + \mathbf{C}$$

and, so, $\mathbf{C} = \langle 0, 0, 1\rangle$. This gives

$$\mathbf{r}(t) = \langle -2\cos(t^2), 2\sin(t^2), t^2 + 1\rangle.$$

(c) Compute the arc length function for the vector-valued function measured from $t = 0$ and reparameterize the curve found in part (b), using this arc length function, $s(t)$.
Solution: The arc length function for $\mathbf{r}(t)$ is defined by

$$s(t) = \int_0^t |\mathbf{v}(u)|du.$$

We first compute the length of the velocity function $\mathbf{v}(u)$:

$$|\mathbf{v}(u)| = \sqrt{16u^2\sin^2(u^2) + 16u^2\cos^2(u^2) + 4u^2} = \sqrt{20u^2} = 2\sqrt{5}u.$$

Then, integrating this from 0 to t, the arc length function is $s(t) = \sqrt{5}t^2$.
To reparametrize the curve by arc length, we substitute $t^2 = s/\sqrt{5}$ in $\mathbf{r}(t)$:

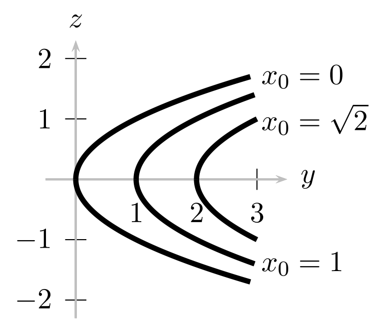
$$\mathbf{r}(s) = \left\langle -2\cos\left(\frac{s}{\sqrt{5}}\right), 2\sin\left(\frac{s}{\sqrt{5}}\right), \frac{s}{\sqrt{5}} + 1\right\rangle.$$

4. (12 points) Consider the function $f(x, y) = \sqrt{y - x^2}$.

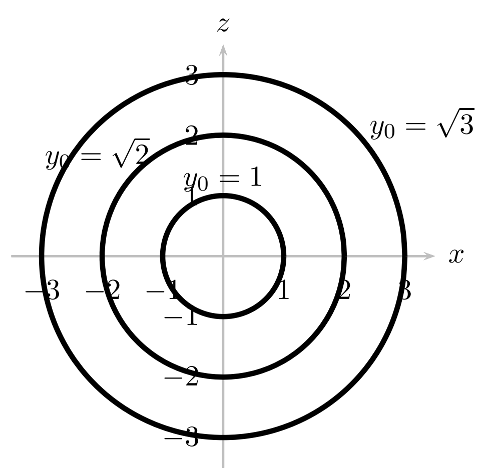
(a) Find the domain and range of the function.
Solution: In order for the square-root to be well-defined, we need $y - x^2 \geq 0$. This means that the domain of $f(x, y)$ is the set of points $\{(x, y) \mid y \geq x^2\}$. We know the range cannot be anything more than $[0, \infty)$, since this is the range that the square-root function can take. Taking $x = 0$, we see that this function's range is at least as big as that of $\sqrt{(}y)$. These together tell us that the range of $f(x, y)$ is $[0, \infty)$.

(b) Sketch the surface corresponding to the square of this function, $z^2 = y - x^2$. To get full credit, you must show all of your work, including a few 2-dimensional sketches of the traces of the surface.

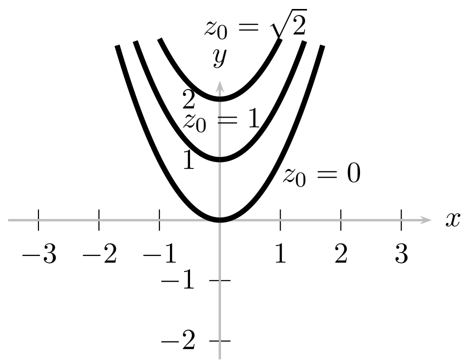
Solution: Rewrite the equation as $x^2 + z^2 = y$. Then, for fixed x value, $x = x_0$, we have $y = z^2 + x_0^2$. For $x_0 = 0$, $y = z^2$ gives a parabola through the origin in the $yz-$plane. For other x_0, the parabola goes through $(x_0^2, 0)$. The traces look like:



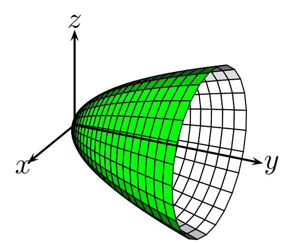
For fixed y value, y_0, there are no traces if $y_0 < 0$. If $y_0 = 0$, then the trace is a point $(0, 0)$ in the $xz-$plane. If $y_0 > 0$, then the trace is a circle of radius $\sqrt{y_0}$ in the $xz-$plane.



For fixed z value, z_0, the traces are also parabolas in the $xy-$plane.



Putting these together, we get:



(c) Give the name of the surface drawn in part (b).
Solution: This is an elliptic paraboloid.

5. (12 points) Find all first partial derivatives of $f(x, y) = x \ln(xy + 1)$
Solution: Using standard differentiation rules, the partial derivatives are

$$\frac{\partial f}{\partial x} = \ln(xy + 1) + \frac{xy}{xy + 1} \text{ and}$$
$$\frac{\partial f}{\partial y} = \frac{x^2}{xy + 1}.$$

6. (12 points) Given the function $f(x, y, z) = x^3 - 2y^2 + 2z^2 - 4$,

(a) Find the gradient of $f(x, y, z)$ at the point $P(1, -1, 3)$.
Solution:

$$\begin{aligned} \nabla f(x, y, z) &= \langle f_x, f_y, f_z \rangle \\ &= \langle 3x^2, -4y, 4z \rangle. \end{aligned}$$
$$\text{So } \nabla f(1, -1, 3) = \langle 3, 4, 12 \rangle$$

(b) What is the direction of the maximum increase in the value of $f(x, y, z)$ at P?
Solution: The direction of maximum increase of $f(x, y, z)$ at P is given by the unit vector in the direction of $\nabla f(x, y, z)$ at P.
The length of the gradient of $f(x, y, z)$ at $P(1, -1, 3)$ is $|\nabla f(1, -1, 3)| = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$. Therefore, the unit vector in the direction of the maximum increase of f at P is $\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \rangle$.

(c) What is the derivative of $f(x, y, z)$ in the direction of $\mathbf{u} = \langle \frac{4}{5}, \frac{3}{5}, 0 \rangle$ at P?
Solution: First, check that $\mathbf{u}$ a unit vector: $|\mathbf{u}| = \sqrt{(\frac{4}{5})^2 + (\frac{3}{5})^2 + 0} = 1$, so it is a unit vector. Then, the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(1, -1, 3) = \nabla f(1, -1, 3) \cdot \mathbf{u} = \langle 3, 4, 12 \rangle \cdot \left\langle \frac{4}{5}, \frac{3}{5}, 0 \right\rangle = \frac{24}{5}.$$

7. (10 points) Let $w(x, y, z) = \ln(x + y + z)$, where $x(s, t) = st$, $y(s, t) = s + t$, and $z(s, t) = s - t$. Find $\frac{\partial w}{\partial s}$, expressed as a function of s and t only.
Solution: Applying the Chain Rule, we obtain the partial derivative:

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= \frac{t}{x + y + z} + \frac{1}{x + y + z} + \frac{1}{x + y + z} \\ &= \frac{t + 2}{x + y + z} = \frac{t + 2}{\underbrace{(st)}_{x} + \underbrace{(s + t)}_{y} + \underbrace{(s - t)}_{z}} \\ &= \frac{t + 2}{s(t + 2)} = \frac{1}{s}. \end{aligned}$$

8. (12 points) Given the function $f(x, y) = (x^2 - 4)(y + 3)$,

(a) Find all critical points of $f(x, y)$
Solution: In order to find the critical points, we compute and set the first partial derivatives of $f(x, y)$ equal to zero. Computing, we get

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x(y + 3), \\ \frac{\partial f}{\partial y} &= x^2 - 4. \end{aligned}$$

From here, we see that $\frac{\partial f}{\partial y} = 0$ only when $x = \pm 2$. For these values of x, $\frac{\partial f}{\partial x} = 0$ only when $y = -3$. Thus, $f(x, y)$ has two critical points: (2,-3), (-2,-3).

(b) Use the Second Derivative Test to determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or saddle point.
Solution: From above, $f(x, y)$ has two critical points: (2,-3), (-2,-3). Computing the second derivatives of $f(x, y)$, we have $f_{xx} = 2(y + 3)$, $f_{yy} = 0$, and $f_{xy} = 2x$. So, both $f_{xx}(\pm 2, -3) = f_{yy}(\pm 2, -3) = 0$, while $f_{xy}(\pm 2, -3) = \pm 4$. Computing $D(x, y) = f_{xx}(x, y) f_{yy}(x, y) 0 - f_{xy}^2(x, y)$ at the two critical points, we find $D(\pm 2, -3) = -16$. Since this is negative, both critical points are saddle points.

9. (12 points) A rectangle with sides parallel to the $x-$ and $y-$axes is inscribed in the region in the first quadrant bounded by the axes and the line $x + 2y = 2$. Using Lagrange Multipliers, find the maximum area of this rectangle. No credit will be given for solutions that do not use Lagrange Multipliers.

Solution: Let x be the length of the base of the rectangle and y be its height. Since the rectangle is in the first quadrant, we know that x and y are both positive.

We now want to maximize the function $A(x, y) = xy$, subject to the constraint $g(x, y) = x + 2y - 2 = 0$. Computing the first partial derivatives of $A(x, y)$, we find a single critical point at $(0, 0)$, but this gives a rectangle of zero area (which won't have the maximum area). So, to find the rectangle with maximum area, we use the method of Lagrange multipliers. The gradients of $A(x, y)$ and $g(x, y)$ are:

$$\nabla A(x, y) = \langle y, x \rangle,$$
$$\text{and } \nabla g(x, y) = \langle 1, 2 \rangle.$$

From the method of Lagrange multipliers, we then get three equations to solve:

$$y = \lambda \cdot 1 \tag{1}$$
$$x = \lambda \cdot 2 \tag{2}$$
$$g(x, y) = x + 2y - 2 = 0 \tag{3}$$

Substituting Equations (1) and (2) into (3), we get $2\lambda + 2\lambda - 2 = 0$, or $\lambda = \frac{1}{2}$. Plugging this value for λ back into (1) and (2), we find $x = 1$ and $y = \frac{1}{2}$. The area of the rectangle with this base and height is $A(1, \frac{1}{2}) = \frac{1}{2}$.

End of Exam