Math 13 | Tufts University | December 15, 2010

Department of Mathematics

Final Examination

No calculators, notes, scratch paper or books are allowed. You must show all your work in your blue book in order to receive full credit. A correct answer with no work may not necessarily score any points. **Simplify** *your answers as much as possible. Cross out any work you do not want graded. Sign your exam book, indicating that you have neither given nor received help during this exam. Any violations will be reported to the appropriate dean, and will result in an F for the course.*

1. (10 points): **True** or **False** - No Partial Credit: Answer the following as True or False. (Only answers will be graded.)

   (a) $\text{comp}_{\vec{u}}\vec{v}$ is positive where $\vec{v}$ and $\vec{u}$ are shown in Figure 1. ($\text{comp}_{\vec{u}}\vec{v}$ is the scalar projection of $\vec{v}$ onto $\vec{u}$)

   (b) $\vec{u} \times \vec{v}$ points into the page, where $\vec{v}$ and $\vec{u}$ are shown in Figure 1.

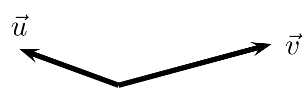

Figure 1: Problem 1(a), 1(b)

   (c) $\int_C \mathbf{F} \cdot d\mathbf{r} < 0$ where $\mathbf{F}$ is the vector field and C is the curve shown in Figure 2.

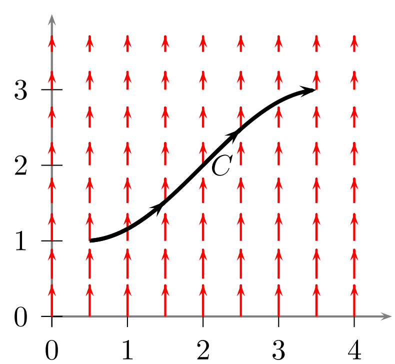

Figure 2: Problem 1(c)

   (d) If the maximum value of $f(x, y, z)$ on the level surface $g(x, y, z) = k$ occurs at $(x, y, z) = (a, b, c)$ then $\nabla f(a, b, c)$ is orthogonal to $\nabla g(a, b, c)$.

   (e) $\int_C \nabla f \cdot d\mathbf{r} = 0$ for a closed curve C and differentiable function f.

Exam Continues on Other Side

2. (10 points): Consider the plane containing the point $(1, 1, 1)$ and the line given by the parametric equations

$$\begin{aligned} x &= 1 + t \\ y &= 2 - 3t \\ z &= 3 + 2t. \end{aligned}$$

Give an equation for this plane.

3. (10 points): Give an equation for the plane tangent to the surface $x^3 - 2xyz + y^2 + z^2 = 20$ at the point $(3, 2, 1)$.

4. (10 points): The function

$$f(x, y) = 6y^2 - 2y^3 + 3x^2 + 6xy$$

has critical points at $(0, 0)$ and at $(-1, 1)$. Classify each as a local minimum, local maximum, or neither (i.e., saddle point).

5. (15 points): Let $\mathcal{E}$ be the region in the first octant under the graph $z = 4 - x^2 - y$ (Figure 3). In the figure, we have indicated the equations of the curves in which this surface intersects each of the coordinate planes:

$$\begin{aligned} xy-\text{plane } (z = 0)&: \quad x^2 + y = 4 \\ xz-\text{plane } (y = 0)&: \quad x^2 + z = 4 \\ yz-\text{plane } (x = 0)&: \quad y + z = 4 \end{aligned}$$

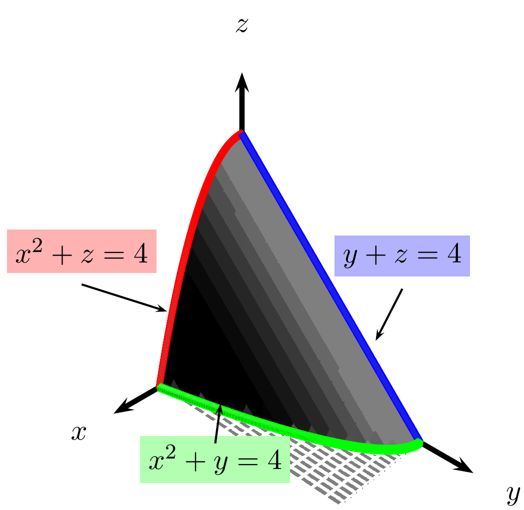

Figure 3: $z = 4 - x^2 - y$

(a) Set up an iterated triple integral expressing the volume of $\mathcal{E}$, in the order $dz\ dy\ dx$.

(b) Set up an iterated triple integral expressing the volume of $\mathcal{E}$, in the order $dx\ dy\ dz$.

(c) Calculate the volume.

6. (15 points): The vector field

$$\boldsymbol{F}(x,y,z) = (y + 2xy\sin z)\vec{i} + (x + x^2\sin z)\vec{j} + (1 + x^2 y\cos z)\vec{k}$$

is conservative.

(a) Find a potential function for $\boldsymbol{F}$.

(b) Evaluate the line integral $\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r}$, where $\mathcal{C}$ is the curve with parametric equations

$$x = t^2$$
$$y = t^3$$
$$z = \frac{\pi}{2}t$$

for $-1 \le t \le 1$.

7. (10 points): Evaluate the line integral

$$\oint_{\mathcal{C}} (x^3 + 2xy + y^3)dx + (x^2 + 3xy^2 + 5x)dy$$

where $\mathcal{C}$ is the circle $x^2 + y^2 = 1$, oriented counterclockwise.

8. (10 points): Use Stokes' Theorem to calculate the integral

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$$

where $\mathbf{F}(x,y,z) = (x^2 - z^2)\vec{i} + y\vec{j} + 2xz\vec{k}$ and $\mathcal{C}$ is the boundary of the triangle with vertices $(4,0,0)$, $(0,4,0)$, $(0,0,4)$, traversed counterclockwise when seen from above (Figure 4).

(Note that this triangle lies in the plane $z = 4 - x - y$.)

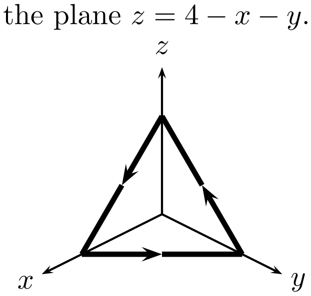

Figure 4: Problem 8

Exam Continues on Other Side

9. (10 points): Calculate the flux integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = x^2\vec{i} + 2yz\vec{j} + y^2z\vec{k}$ and $\mathcal{S}$ is the surface of the cube with faces in the coordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$, with outward orientation. (*Hint:* Use the Divergence Theorem.)

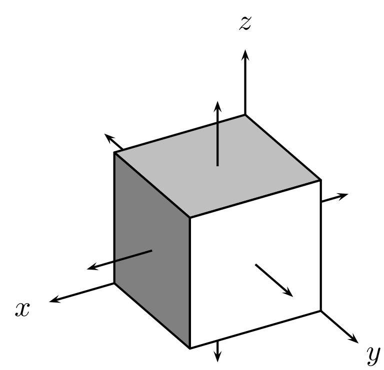

Figure 5: Problem 9

End of Exam