Department of Mathematics

Exam 2 Solutions

1. (15 points) *Find and classify (as local minima, local maxima, or saddle points) all critical points of the function* $g(x, y) = x^4 - 2xy + \frac{1}{2}y^2$.

We have the partial derivatives

$$g_x(x,y) = 4x^3 - 2y \qquad g_{xy}(x,y) = -2 \qquad g_y(x,y) = -2x + y$$
$$g_{xx}(x,y) = 12x^2 \qquad g_{yy}(x,y) = 1.$$

The critical points occur at $(x, y)$ where

$$\begin{aligned} g_x(x,y) &= 0 \quad \text{and} & g_y(x,y) &= 0 \\ 4x^3 - 2y &= 0 & -2x + y &= 0 \\ y &= 2x^3 & & \\ & & \text{so:} \quad -2x + 2x^3 &= 0 \\ & & x(x^2-1) &= 0 \\ & & x &= 0, \pm 1 \\ y &= 0, \pm 2. & & \end{aligned}$$

The critical points are $(0,0)$, $(1,2)$, $(-1,-2)$. Compute $D = g_{xx}(x,y)g_{yy}(x,y) - [g_{xy}(x,y)]^2$ at each critical point:

$$\begin{aligned} (0,0):& \quad D = 0 \cdot 1 - [-2]^2 = -4 < 0 \\ (1,2):& \quad D = 12 \cdot 1 - [-2]^2 = 8 > 0 \quad \text{and} \quad g_{xx}(1,2) = 12 > 0 \\ (-1,-2):& \quad D = 12 \cdot 1 - [-2]^2 = 8 < 0 \quad \text{and} \quad g_{xx}(-1,-2) = 12 > 0 \end{aligned}$$

By the second partials test, $(0,0)$ is a saddle point, and $(1,2)$ and $(-1,-2)$ are local minima.

2. (15 points) *Use the method of Lagrange multipliers to find the maximum volume of a rectangular box without a top that can be made of 12 square meters of material.*

Let the box have dimensions $x$, $y$, and $z$. The volume of the box is $f(x,y,z) = xyz$ and the surface area $g(x,y,z) = xy + 2xz + 2yz$. The constraint is $g(x,y,z) = 12\ \text{m}^2$. Then

$$\nabla f = (yz, xz, xy) \quad \text{and} \quad \nabla g = (y + 2z, x + 2z, 2x + 2y)$$

and we solve $\nabla f = \lambda \nabla g$ (this is the method of Lagrange multipliers). This generates three equations (one for each component of the gradients)

$$\begin{aligned} yz &= \lambda(y+2z) & xz &= \lambda(x+2y) & xy &= \lambda(2x+2y) \\ \frac{1}{\lambda} &= \frac{y+2z}{yz} & \frac{1}{\lambda} &= \frac{x+2z}{xz} & \frac{1}{\lambda} &= \frac{2x+2y}{xy} \\ &= \frac{1}{z} + \frac{2}{y} & &= \frac{1}{z} + \frac{2}{x} & &= \frac{2}{y} + \frac{2}{x} \end{aligned}$$

which can be solved by recognizing that $1/\lambda$ is common to all three. The first two give $x = y$ and the second two give $y = 2z$, so $x = 2z$ as well. Then the constraint is

$$\begin{aligned} 12 &= g(2z, 2z, z) = 4z^2 + 4z^2 + 4z^2 = 12z^2 \\ z^2 &= 1 \\ z &= \pm 1 \end{aligned}$$

and we discard the negative solution as unphysical. Thus $z = 1$ and $x = y = 2$. The volume is maximized at $(x, y, z) = (2, 2, 1)$. The maximum value is $f(2, 2, 1) = 4\ \text{m}^3$.

3. (10 points) *Evaluate the integral* $\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{1+y^3}\, dy\, dx$ *by reversing the order of integration.*

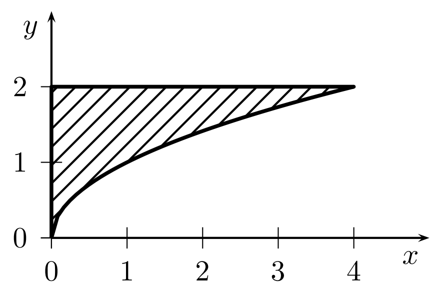


Figure 1: Domain for problem 3

$$\begin{aligned} \int_0^4 \int_{\sqrt{x}}^2 \frac{1}{1+y^3}\, dy\, dx &= \int_0^2 \int_0^{y^2} \frac{1}{1+y^3}\, dx\, dy \\ &= \int_0^2 \frac{1}{1+y^3} [x]_0^{y^2}\, dy \\ &= \int_0^2 \frac{1}{1+y^3} y^2\, dy && u = 1 + y^3 \\ &= \frac{1}{3} \int_1^9 \frac{1}{u}\, du && du = 3y^2\, dy \\ &= \frac{1}{3} [\ln|u|]_1^9 = \frac{1}{3}(\ln 9 - \ln 1) = \frac{1}{3} \ln 9 \end{aligned}$$

4. (15 points) *Figure 2 shows the solid bounded by the surfaces*

$$x = 0, \quad y = z^2, \quad z = 2, \quad \text{and} \quad y = 2x$$

*as well as the projection of the solid onto the xy-plane.* **Set up** *iterated triple integrals that yield the volume of the solid in the following orders.* **Do not evaluate!**

(a) $dz\ dy\ dx$

(b) $dx\ dy\ dz$

(a) Examining the volume, the $z$-integral goes from the lower surface $y = z^2$ to the upper surface $z = 2$. Examining the shaded region in the $x$-$y$ plane, the $y$-integral goes from

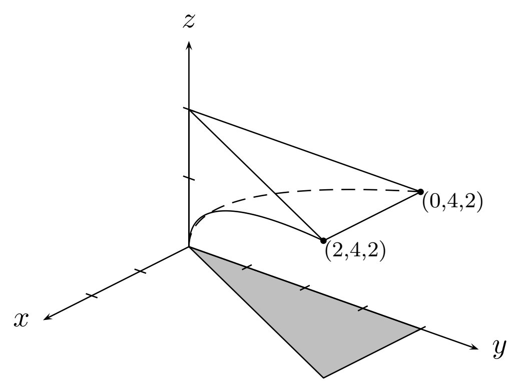


Figure 2: Problem 4

the line $y = 2x$ to the line $y = 4$. Examining the projection of the shaded region onto the $x$-axis, the $x$-integral has bounds $x = 0$ and $x = 2$.

Thus we find $V = \int_0^2 \int_{2x}^4 \int_{\sqrt{y}}^2 1\, dz\, dy\, dx$

(b) Examining the volume, the $x$-integral goes from the back surface $x = 0$ to the front surface (plane) $y = 2x$. Examining the region in the $y$-$z$ plane, the $y$-integral goes from the line $y = 0$ to the curve $y = z^2$. Examining the $z$-axis, the $z$-integral has bounds $z = 0$ and $z = 2$.

Thus we find $V = \int_0^2 \int_0^{z^2} \int_0^{y/2} 1\, dx\, dy\, dz$

5. (15 points) *Let $E$ be the region in the first octant bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 32$. See Figure 3 below. Consider the integral*

$$\iiint_E xz\, dV.$$

*(a) Rewrite the integral as an iterated integral in* rectangular coordinates.

*(b) Rewrite the integral as an iterated integral in* cylindrical coordinates.

*(c) Rewrite the integral as an iterated integral in* spherical coordinates.

(*Note:* ***do not evaluate*** *any of the above.*)

(a) We set up the rectangular integral in the order $dz\, dy\, dx$. The lower surface is $z = \sqrt{x^2 + y^2}$ and the upper surface is $z = \sqrt{32 - x^2 - y^2}$ so these are the bounds on the $z$-integral. The two surfaces meet where their $z$-values agree:

$$\begin{aligned} x^2 + y^2 &= 32 - x^2 - y^2 \\ 2x^2 + 2y^2 &= 32 \\ x^2 + y^2 &= 16 \end{aligned}$$

which is a circle of radius 4, as drawn in Figure 3 (shaded region). We integrate over this region by having $y$ range from zero to $\sqrt{16 - x^2}$ and $x$ from 0 to 4. That is:

$$\iiint_E xz\, dV = \int_0^4 \int_0^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} xz\, dz\, dy\, dx.$$

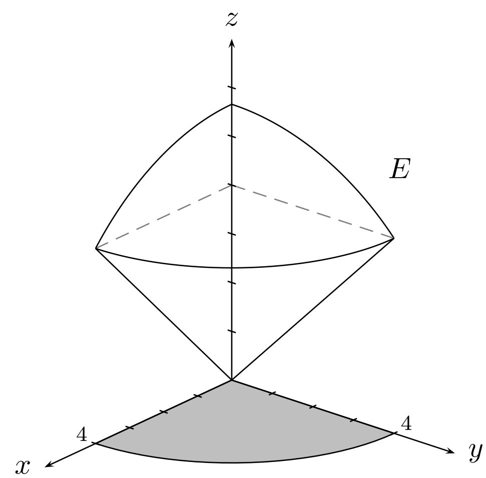


Figure 3: Problem 5

(b) Using the standard orientation for cylindrical co-ordinates, $r^2 = x^2 + y^2$ and the region in the plane can be expressed as $0 < \theta < \pi/2$ and $0 < r < 4$. The differential volume element becomes $dV = r\ dz\ dr\ d\theta$. Changing variables in the integrand then gives

$$\iiint_E xz\ dV = \int_0^{\pi/2} \int_0^4 \int_r^{\sqrt{32-r^2}} (r\cos\theta) z\ r\ dz\ dr\ d\theta$$
$$= \int_0^{\pi/2} \int_0^4 \int_r^{\sqrt{32-r^2}} zr^2 \cos\theta\ dz\ dr\ d\theta.$$

(c) We need to find an equation for the cone in spherical co-ordinates:

$$\begin{aligned} z^2 &= x^2 + y^2 \\ 2z^2 &= x^2 + y^2 + z^2 \\ 2(\rho\cos\phi)^2 &= \rho^2 \\ \cos^2\phi &= 1/2 \\ \cos\phi &= 1/\sqrt{2} = \sqrt{2}/2 \\ \phi &= \pi/4. \end{aligned}$$

The bound on $\theta$ remains as in part (b), $\rho$ extends from the origin to the sphere $\rho = \sqrt{32} = 4\sqrt{2}$, and now the differential volume is $dV = \rho^2 \sin\phi\ d\phi\ d\rho\ d\theta$. Changing variables in the integrand then gives

$$\iiint_E xz\ dV = \int_0^{\pi/2} \int_0^{4\sqrt{2}} \int_0^{\pi/4} (\rho\sin\phi\cos\theta)(\rho\cos\phi)\ \rho^2 \sin\phi\ d\phi\ d\rho\ d\theta$$
$$= \int_0^{\pi/2} \int_0^{4\sqrt{2}} \int_0^{\pi/4} \rho^4 \cos\theta \sin^2\phi \cos\phi\ d\phi\ d\rho\ d\theta.$$

6. (15 points) *Evaluate the integral* $\displaystyle\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x\ dz\ dy\ dx.$

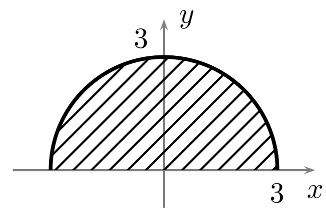


Figure 4: Domain for Problem 6

The domain in the $x$-$y$ plane is given in Figure 4, which we change to cylindrical co-ordinates:

$$\begin{aligned}
\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x\,dz\,dy\,dx &= \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} r\cos\theta\; r\,dz\,dr\,d\theta \\
&= \int_{0}^{\pi}\cos\theta\,d\theta\;\int_{0}^{3}\int_{0}^{9-r^2} r^2\,dz\,dr \\
&= [\,\sin\theta\,]_{0}^{\pi}\;\int_{0}^{3}\int_{0}^{9-r^2} r^2\,dz\,dr \\
&= (0-0)\;\int_{0}^{3}\int_{0}^{9-r^2} r^2\,dz\,dr \\
&= 0.
\end{aligned}$$

7. (15 points) *Evaluate the following line integrals.*

(a) $\int\limits_C xy^3\,ds$ *where $C$ is the quarter circle of radius 2 shown in Figure 5.*

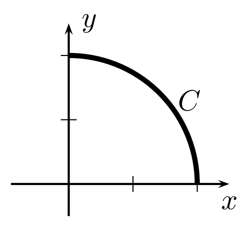


Figure 5: Problem 7a

Parametrize the curve for $0 \le t \le \pi/2$

$$x(t) = 2\cos t \qquad\qquad y(t) = 2\sin t$$
$$x'(t) = -2\sin t \qquad\qquad y'(t) = 2\cos t$$

and use that

$$\begin{aligned}
ds &= \left(\left[x'(t)\right]^2 + \left[y'(t)\right]^2\right)^{1/2}\,dt \\
&= \left([-2\sin t]^2 + [2\cos t]^2\right)^{1/2}\,dt \\
&= \left(4\sin^2 t + 4\cos^2 t\right)^{1/2}\,dt \\
&= 2\,dt
\end{aligned}$$

to convert the integral to

$$\int_C xy^3\,ds = \int_0^{\pi/2} x(t)\,[y(t)]^3\;2\,dt$$

$$= \int_0^{\pi/2} 2\cos t\,[2\sin t]^3\;2\,dt$$

$$= 32\int_0^{\pi/2} \cos t\sin^3 t\,dt \qquad u = \sin t$$

$$= 32\int_0^{1} u^3\,du \qquad du = \cos t\,dt$$

$$= 32\left[\frac{1}{4}u^4\right]_0^1 = 8(1-0) = 8.$$

(b) $\int_C \mathbf{F}(x,y,z)\cdot d\boldsymbol{r}$ *where* $\mathbf{F}(x,y,z) = (x+y)\,\mathbf{i} - x\,\mathbf{j} + \sin^2 x\,\mathbf{k}$ *and* $C$ *is the curve parametrized by* $\boldsymbol{r}(t) = t\,\mathbf{i} - t^2\,\mathbf{j} + \cos^2 t\,\mathbf{k}$ *for* $0 \le t \le \dfrac{\pi}{2}$.

We need to find $\mathbf{r}'(t)$ and $\mathbf{F}(\mathbf{r}(t))$:

$$\mathbf{r}(t) = (x(t),\; y(t),\; z(t)) = \left(t,\; -t^2,\; \cos^2 t\right) \qquad \mathbf{F}(\mathbf{r}(t)) = \left(x(t)+y(t),\; -x(t),\; \sin^2 x(t)\right) = \left(t - t^2,\; -t,\; \sin^2 t\right)$$

$$\mathbf{r}'(t) = (1,\; -2t,\; -2\cos t\sin t)\,.$$

Then the integral becomes

$$\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \int_0^{\pi/2} \left(t - t^2,\; -t,\; \sin^2 t\right)\cdot(1,\; -2t,\; -2\cos t\sin t)\;dt$$

$$= \int_0^{\pi/2} \left(t - t^2 + 2t^2 - 2\cos t\sin^3 t\right)\;dt$$

$$= \int_0^{\pi/2} \left(t + t^2\right)\;dt - 2\int_0^{\pi/2} \cos t\sin^3 t\,dt \qquad u = \sin t$$

$$= \left[\frac{1}{2}t^2 + \frac{1}{3}t^3\right]_0^{\pi/2}\;dt - 2\int_0^{1} u^3\,du \qquad du = \cos t\,dt$$

$$= \left(\frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \frac{1}{3}\left(\frac{\pi}{2}\right)^3\right) - 2\left[\frac{1}{4}u^4\right]_0^1 = \frac{\pi^2}{8} + \frac{\pi^3}{24} - \frac{1}{2}.$$

**End of Exam**