Math 13 | Tufts University | October 11, 2010

Department of Mathematics

Exam 1 Solutions

1. (15 points): *Consider the triangle whose vertices are*

$$P(1,2,0) \quad Q(2,1,2) \quad R(4,2,1).$$

   (a) *Give parametric equations for the line containing the edge PQ*:
   A direction vector would be

$$\vec{v} = \vec{PQ} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$$

   so, using $P_0 = P$,

$$\begin{cases} x &= 1 & +t \\ y &= 2 & -t \\ z &= & 2t. \end{cases}$$

   (b) *Give an equation for the plane containing the triangle* $\triangle PQR$:
   To get a normal vector to the plane, we use

$$\begin{aligned} \vec{PQ} \times \vec{PR} &= (\vec{\imath} - \vec{\jmath} + 2\vec{k}) \times (3\vec{\imath} + \vec{k}) \\ &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -1 & 2 \\ 3 & 0 & 1 \end{vmatrix} \\ &= \vec{\imath}(-1-0) - \vec{\jmath}(1-6) + \vec{k}(0-(-3)) \\ &= -\vec{\imath} + 5\vec{\jmath} + 3\vec{k}; \end{aligned}$$

   it might be more convenient to use its negative:

$$\vec{n} = \vec{\imath} - 5\vec{\jmath} - 3\vec{k}.$$

   Then the equation of the plane, using $P_0 = P$, is

$$(x-1) - 5(y-2) - 3(z-0) = 0$$

   or

$$x - 5y - 3z = -9.$$

   (c) *Find the area of the triangle* $\triangle PQR$:
   We know that the area of the *parallelogram* with sides $\vec{PQ}$ and $\vec{PR}$ has area

$$\begin{aligned} |\vec{PQ} \times \vec{PR}| &= |-\vec{\imath} + 5\vec{\jmath} + 3\vec{k}| \\ &= \sqrt{1+25+9} \\ &= \sqrt{35} \end{aligned}$$

   so the area of the *triangle* with these sides is half as much

$$A = \frac{\sqrt{35}}{2}.$$

2. (15 points): Let

$$\vec{u} = \vec{\imath} + \vec{\jmath} - \vec{k}, \quad \vec{v} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}.$$

Calculate

(a) *The cosine of the angle between* $\vec{u}$ *and* $\vec{v}$:

$$\begin{aligned} \cos\theta &= \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \\ &= \frac{-2}{(\sqrt{3})(\sqrt{14})} \\ &= -\frac{2}{\sqrt{42}}. \end{aligned}$$

(b) *The scalar projection* $\text{comp}_{\vec{v}}\,\vec{u}$ *of* $\vec{u}$ *on* $\vec{v}$:

$$\begin{aligned} \text{comp}_{\vec{v}}^{\vec{u}} &= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} \\ &= \frac{2-1-3}{\sqrt{4+1+9}} = -\frac{2}{\sqrt{14}}. \end{aligned}$$

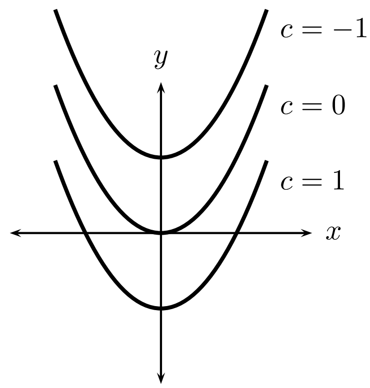
(c) *The vector projection* $\text{proj}_{\vec{u}}\,\vec{v}$ *of* $\vec{v}$ *on* $\vec{u}$:

$$\begin{aligned} \text{proj}_{\vec{u}}^{\vec{v}} &= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} \\ &= \frac{-2}{3}(\vec{\imath} + \vec{\jmath} - \vec{k}) \\ &= -\frac{2}{3}\vec{\imath} - \frac{2}{3}\vec{\jmath} + \frac{2}{3}\vec{k}. \end{aligned}$$

3. (10 points): Let

$$f(x, y) = x^2 - y.$$

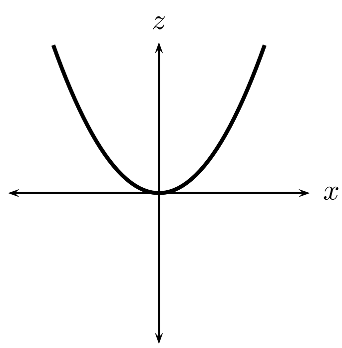
(a) *Sketch and label the level curves of f corresponding to the values $f = -1, 0,$ and 1.*:
$x^2 - y = c$ can be rewritten $y = x^2 - c$, so the three curves are upward-opening parabolas:



(b) *In two separate pictures, sketch the traces of the graph $z = f(x, y)$ in the (x, z) and (y, z) planes*:

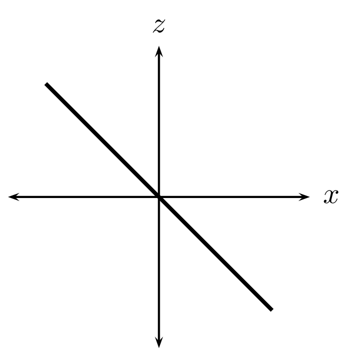
i. The (x, z) plane corresponds to $y = 0$, and the trace is given by

$$z = x^2$$



ii. The (y, z) plane corresponds to $x = 0$, and the trace is given by

$$z = -y :$$



4. (15 points): *A particle moves with acceleration*

$$\vec{a}(t) = (t+1)\vec{\imath} + e^{2t}\vec{\jmath} + \pi \sin \pi t \vec{k}$$

initial velocity

$$\vec{v}(0) = \vec{\imath} + \vec{\jmath}$$

and initial position

$$\vec{r}(0) = \vec{\imath} + \vec{\jmath} + \vec{k}.$$

(a) *Give a formula for its position at any later time t (that is, find the position function):*

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)\ dt \\ &= \left(\frac{t^2}{2} + t\right)\vec{\imath} + \frac{1}{2}e^{2t}\vec{\jmath} - \cos \pi t + \vec{C};\end{aligned}$$

matching the initial velocity

$$\vec{\imath} + \vec{\jmath} = 0\vec{\imath} + \frac{1}{2}\vec{\jmath} - \vec{k} + \vec{C}$$

yields

$$\vec{C} = \vec{\imath} + \frac{1}{2}\vec{\jmath} + \vec{k}$$

so

$$\vec{v}(t) = \left(\frac{t^2}{2} + t + 1\right)\vec{\imath} + \left(\frac{1}{2}e^{2t} + \frac{1}{2}\right)\vec{\jmath} + (1 - \cos \pi t)\,\vec{k};$$

integrating again, we have

$$\vec{r}(t) = \int \vec{v}(t)\ dt = \left(\frac{t^3}{6} + \frac{t^2}{2} + t\right)\vec{\imath} + \left(\frac{1}{4}e^{2t} + \frac{t}{2}\right)\vec{\jmath} + \left(t - \frac{1}{\pi}\sin \pi t\right)\vec{k} + \vec{C}$$

and matching the initial position

$$\vec{\imath} + \vec{\jmath} + \vec{k} = 0\vec{\imath} + \frac{1}{4}\vec{\jmath} + 0\vec{k} + \vec{C}$$

yields

$$\vec{C} = \vec{\imath} + \frac{3}{4}\vec{\jmath} + \vec{k}$$

so

$$\vec{r}(t) = \left(\frac{t^3}{6} + \frac{t^2}{2} + t + 1\right)\vec{\imath} + \left(\frac{1}{4}e^{2t} + \frac{t}{2} + \frac{3}{4}\right)\vec{\jmath} + \left(t - \frac{1}{\pi}\sin \pi t + 1\right)\vec{k}.$$

(b) *Suppose w is related to the position of the particle via*

$$w = \frac{x^2 z^3}{y}.$$

Find $\frac{dw}{dt}$ when $t = 0$:
Differentiating

$$w = \frac{x^2 z^3}{y}$$

with respect to t yields

$$\frac{dw}{dt} = \left(\frac{2xz^3}{y}\right)\frac{dx}{dt} + \left(-\frac{x^2 z^3}{y^2}\right)\frac{dy}{dt} + \left(\frac{3x^2 z^2}{y}\right)\frac{dz}{dt}.$$

When $t = 0$, we have

$$(x, y, z) = \vec{r}(0) = (1, 1, 1)$$

and

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \vec{v}(0) = (1, 1, 0)$$

so

$$\frac{dw}{dt} = \left(\frac{2 \cdot 1 \cdot 1^2}{1}\right) \cdot 1 + \left(-\frac{1^2 \cdot 1^3}{1^2}\right) \cdot 1 + \left(\frac{3 \cdot 1^2 \cdot 1^2}{1}\right) \cdot 0 = 2 - 1 + 0 = 1.$$

5. (5 points): *Consider the lines given parametrically by:*

$$L_1: \begin{cases} x &= 1 &+2t \\ y &= -1 &+3t \\ z &= 2 &-t \end{cases} \qquad L_2: \begin{cases} x &= -1 &+t \\ y &= -4 &+2t \\ z &= 3 &+3t \end{cases}$$

Either show that these lines don't intersect, or give the coordinates of a point of intersection.
:

To check for intersection, we use a different letter for the two "time" parameters, and set corresponding coordinates equal:

$$\begin{aligned} 1 + 2t &= -1 + s \\ -1 + 3t &= -4 + 2s \\ 2 - t &= 3 + 3s. \end{aligned}$$

If we move all the constants to the left side, we get

$$\begin{aligned} 2 + 2t &= s \\ 3 + 3t &= 2s \\ -1 - t &= 3s. \end{aligned}$$

Solving each for s in terms of t, we get

$$\begin{aligned} s &= 2(t+1) \\ s &= \frac{3}{2}(t+1) \\ s &= -\frac{1}{3}(t+1). \end{aligned}$$

The only way that three different multiples of $t+1$ can be equal is if $t+1=0$; thus the solution to our system of equations is

$$\begin{aligned} s &= 0 \\ t &= -1; \end{aligned}$$

substituting either value into the parametric equations for its line yields the point of intersection $(-1,-4,3)$.

Of course, equivalently, we might just have noted that the starting point of the second line is reached by the first line when $t=-1$.

6. (10 points): Give parametric equations for the line of intersection of the two planes

$$2x - y + 3z = 4 \quad \text{and} \quad x - 2y + z = 0:$$

A direction vector for the line is the cross product of the two normal vectors

$$\begin{aligned}\vec{v} = \vec{n_1} \times \vec{n_2} &= (2\vec{\imath} - \vec{\jmath} + 3\vec{k}) \times (\vec{\imath} - 2\vec{\jmath} + \vec{k}) \\ &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -1 & 3 \\ 1 & -2 & 1 \end{vmatrix} \\ &= 5\vec{\imath} + \vec{\jmath} - 3\vec{k}.\end{aligned}$$

We need to find a single point in the intersection of the two planes. You might notice that $(1, 1, 1)$ works; or you can set one coordinate equal to zero. For example, setting $z = 0$ leads to the two equations

$$\begin{aligned}2x - y &= 4 \\ x - 2y &= 0\end{aligned}$$

with solution

$$\begin{aligned}x &= \frac{8}{3} \\ y &= \frac{4}{3}\end{aligned}$$

so another point of intersection is $(\frac{8}{3}, \frac{4}{3}, 0)$. If we use $(1, 1, 1)$, we get the equations for the line

$$\begin{cases} x = 1 + 5t \\ y = 1 + t \\ z = 1 - 3t. \end{cases}$$

7. (10 points): *Consider the function*

$$f(x, y) = \sqrt{x^2 + xy + 2y^2}.$$

Use the linear approximation at $(2, -3)$ *to calculate an approximate value for* $f(2.2, -3.1)$:

The two first-order partial derivatives of

$$f(x, y) = \sqrt{4x^2 + y^2}$$

are

$$\begin{aligned}
\frac{\partial f}{\partial x} = f_x(x, y) &= \frac{2x + y}{2\sqrt{x^2 + xy + y^2}} \\
\frac{\partial f}{\partial y} = f_y(x, y) &= \frac{x + 4y}{2\sqrt{x^2 + xy + y^2}};
\end{aligned}$$

at our point $(2, -3)$, these have values

$$\begin{aligned}
f(2, -3) &= \sqrt{2^2 + (2)(-3) + (2)(-3)^2} = \sqrt{4 - 6 + 18} = 4 \\
f_x(2, -3) &= \frac{(2)(2) + (-3)}{4} = \frac{1}{8} \\
f_y(2, -3) &= \frac{2 + (4)(-3)}{4} = \frac{-10}{8} = -\frac{5}{4}
\end{aligned}$$

so the linear approximation to f at $(2, -3)$ is

$$\begin{aligned}
L(2 + \Delta x, -3 + \Delta y) &= f(2, -3) + f_x(2, -3)\Delta x + f_y(2, -3)\Delta y \\
&= 4 + \frac{1}{8}\Delta x - \frac{5}{4}\Delta y.
\end{aligned}$$

the increments in our case are

$$\begin{aligned}
\Delta x &= 2.2 - 2 = 0.2; \\
\Delta y &= -3.1 - (-3) = -0.1
\end{aligned}$$

so

$$\begin{aligned}
L(2.2, -3.1) &= 5 + \frac{1}{8}(0.2) - \frac{5}{4}(-0.1) \\
&= 5 + .025 + 0.125 \\
&= 4.15.
\end{aligned}$$

8. (15 points): *Let*

$$f(x, y, z) = x^2 e^{2y+z}.$$

(a) *Calculate the gradient vector* $\nabla f(3, 1, -2)$ *of* f *at* $(3, 1, -2)$:

$$f_x(x, y, z) = 2xe^{2y+z}$$
$$f_y(x, y, z) = 2x^2 e^{2y+z}$$
$$f_z(x, y, z) = x^2 e^{2y+z}$$

so

$$f_x(3, 1, -2) = 2(3)e^{2(1)+(-2)} = 6$$
$$f_y(3, 1, -2) = 2(3)^2 e^{2(1)+(-2)} = 18$$
$$f_z(3, 1, -2) = (3)^2 e^{2(1)+(-2)} = 9$$

and so

$$\nabla f(3, 1, -2) = 6\vec{\imath} + 18\vec{\jmath} + 9\vec{k}.$$

(b) *Give parametric equations for the plane tangent to the surface*

$$x^2 e^{2y+z} = 9$$

at the point $(3, 1, -2)$:
We need the plane through the point $(3, 1, -2)$ with normal vector $\nabla f(3, 1, -2) = 6\vec{\imath} + 18\vec{\jmath} + 9\vec{k}$; this has equation

$$6(x - 3) + 18(y - 1) + 9(z + 2) = 0$$

or

$$6x + 18y + 9z = 18.$$

9. (10 points): *If x, y, and z are related by*

$$z^4 + x^2yz + 2xy = 83,$$

find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $x = -1$, $y = 2$, and $z = 3$.

Differentiating with respect to x we have

$$4z^3\frac{\partial z}{\partial x} + 2xyz + x^2y\frac{\partial z}{\partial x} + 2y = 0.$$

Plugging in we have

$$4(27)\frac{\partial z}{\partial x} + 2(-1)(2)(3) + (1)(2)\frac{\partial z}{\partial x} + 2(2) = 0$$

or

$$108\frac{\partial z}{\partial x} - 12 + 2\frac{\partial z}{\partial x} + 4 = 0$$

hence

$$\frac{\partial z}{\partial x} = \frac{8}{110}.$$

Differentiating with respect to y we have

$$4z^3\frac{\partial z}{\partial y} + x^2z + x^2y\frac{\partial z}{\partial y} + 2x = 0.$$

Plugging in we have

$$108\frac{\partial z}{\partial y} + 3 + 2\frac{\partial z}{\partial y} - 2 = 0$$

hence

$$\frac{\partial z}{\partial y} = \frac{-1}{110}.$$