Tufts University
Department of Mathematics
Math 13
Final Examination Fall 2009: **Solutions**

Instructions: Please do all problems below. You must show your work and justify your answers in order for partial or full credit to be awarded. No books, notes or calculators are allowed during this exam. *You are required to sign each examination blue book that you are handing in. With your signature, you are pledging that you have neither given nor received any help pertaining to this exam. If you are found in violation of this policy, you will be refered to the Dean of Students and automatically receive an* **F** *for the course.*

Problem 1.

(a) Give an equation for the plane containing the points $(2,2,0)$**,** $(3,0,1)$**, and** $(6,0,0)$**.**

The plane is parallel to the vectors $\langle 1,-2,1\rangle$ and $\langle 4,-2,0\rangle$ hence an example of a normal vector is given by $\langle 2,4,6\rangle = 2\langle 1,2,3\rangle$. Using $\mathbf{n} = \langle 1,2,3\rangle$ we obtain

$$\begin{aligned}\mathbf{n}\cdot\langle x,y,z\rangle &= \mathbf{n}\cdot\langle 2,2,0\rangle\\ x+2y+3z &= 6\end{aligned}$$

which is easily seen to contain the 3 points.

(b) Give an equation for the plane tangent to the surface

$$x^2y - 2xyz + 3z^2 = 3$$

at the point $(-2,-3,-1)$**.**

Define $g(x,y,z) = x^2y - 2xyz + 3z^2$. Then the surface is the level surface $g(x,y,z) = 3$, hence the gradient vector $\nabla g(x,y,z) = \left\langle 2xy - 2yz, x^2 - 2xz, -2xy + 6z\right\rangle$ is everywhere normal to the surface. Thus we may use $\mathbf{n} = \nabla g(-2,-3,-1) = \langle 6,0,-18\rangle$ and obtain

$$\begin{aligned}\langle 6,0,-18\rangle\cdot\langle x,y,z\rangle &= \langle 6,0,-18\rangle\cdot\langle -2,-3,-1\rangle\\ 6x - 18z &= 6\end{aligned}$$

as an equation of our plane.

Problem 2. Find the minimum and maximum values of the function $f(x,y,z) = 8x - 4z$ on the ellipsoid $x^2 + 10y^2 + z^2 = 5$.

Using the method of Lagrange multipliers we set up the system $\nabla f = \lambda\nabla g$ and the constraint $g(x,y,z) = 5$:

$$8 = 2\lambda x \tag{1}$$
$$0 = 20\lambda y \tag{2}$$
$$-4 = 2\lambda z \tag{3}$$
$$5 = x^2 + 10y^2 + z^2 \tag{4}$$

Clearly equations (1) and (3) would be violated if $\lambda = 0$ hence we may assume $\lambda \neq 0$. Thus we obtain from equation (2) that $y = 0$ and by solving (1) and (3) for λ we have

$$\frac{4}{x} = \lambda = \frac{-2}{z}$$

or $x = -2z$. Pluging into (4) we have

$$(-2z)^2 + 10(0)^2 + z^2 = 5$$

with solutions $z = \pm 1$. Hence we have two test points: $(2, 0, -1), (-2, 0, 1)$. We evaluate $f(2, 0, -1) = 20$ and $f(-2, 0, 1) = -20$ hence our maximum is 20 and our minimum is -20.

Problem 3.

(a) Find all the critical points of the function $f(x, y) = x^4 + 32xy + 32y^2$.

The partial derivatives of f are defined and continuous everywhere, hence critical points will only occur when $f_x = f_y = 0$. We have

$$0 = f_x(x, y, z) = 4x^3 + 32y \qquad (1)$$
$$0 = f_y(x, y, z) = 32x + 64y \qquad (2)$$

From (2) we see $y = \frac{-1}{2}x$. Plugging into (1) we have

$$\begin{aligned} 4x^3 - 16x &= 0 \\ 4x(x-2)(x+2) &= 0 \\ x &= 0, \pm 2. \end{aligned}$$

We thus obtain 3 critical points: $(0, 0), (2, -1), (-2, 1)$.

(b) The function $f(x, y) = 8x^3 + y^3 + 6xy$ has a critical point at $(-\frac{1}{2}, -1)$. Determine whether it is a local minimum, a local maximum, or neither, for $f(x, y)$.

We calculate

$$\begin{aligned} f_x &= 24x^2 + 6y \\ f_y &= 3y^2 + 6x \\ f_{xx} &= 48x \\ f_{yy} &= 6y \\ f_{xy} = f_{yx} &= 6 \end{aligned}$$

Hence we evaluate

$$\begin{aligned} D(-.5, -1) &= f_{xx}(-.5, -1) f_{yy}(-.5, -1) - [f_{xy}(-.5, -1)]^2 \\ &= (-24)(-6) - 36 \\ &> 0 \end{aligned}$$

hence at $(-.5, -1)$ the function f has a local max or a local min. Since $f_{xx}(-.5, -1) = -24 < 0$ we conclude that f has a local maximum at $(-.5, -1)$.

Problem 4. Evaluate the line integral $\int_{\mathcal{C}} y\ ds$, where $\mathcal{C}$ is the curve given by the parametric equations

$$\begin{cases} x &= 3t - t^3 \\ y &= 3t^2 \end{cases}, \quad -\sqrt{3} \le t \le \sqrt{3}.$$

After careful calculation we obtain

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \sqrt{(3 - 3t^2)^2 + (6t)^2} dt \\ &= \sqrt{9 - 18t^2 + 9t^4 + 36t^2} dt \\ &= \sqrt{9 + 18t^2 + 9t^4} dt \\ &= \sqrt{(3 + 3t^2)^2} dt \\ &= (3 + 3t^2) dt \end{aligned}$$

since $3 + 3t^2$ is always positive. Hence

$$\begin{aligned} \int_c y ds &= \int_{-\sqrt{3}}^{\sqrt{3}} 3t^2(3 + 3t^2) dt \\ &= 9 \int_{-\sqrt{3}}^{\sqrt{3}} t^2 + t^4 dt \\ &= 9(\frac{t^3}{3} + \frac{t^5}{5})\Big|_{-\sqrt{3}}^{\sqrt{3}} \\ &= 18\sqrt{3} + \frac{162}{5}\sqrt{3} \end{aligned}$$

Problem 5. For each region set up, but do not evaluate, a triple integral in the specified coordinates expressing the given volume

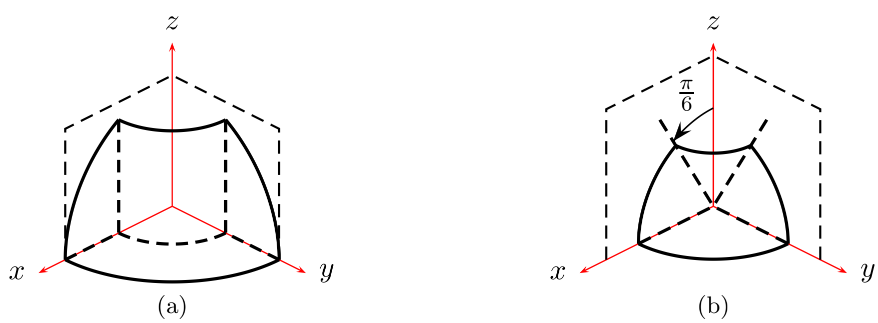

Figure 1: Problem 5

(a) The volume of the region *in the first octant* outside the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$ (see Fig 1a), in cylindrical coordinates.

$$\int_0^{\frac{\pi}{2}} \int_1^2 \int_0^{\sqrt{4-r^2}} 1 \ r \ dz \ dr \ d\theta$$

(b) The volume of the region *in the first octant* inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cone obtained by rotating about the z-axis a ray making angle $\frac{\pi}{6}$ radians with the positive z-axis (see Fig 1b), in spherical coordinates

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} 1 \ \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

Problem 6. Let $\mathcal{E}$ be the pyramidal region with vertices $(0,0,0)$, $(1,0,1)$, $(0,1,1)$, and $(0,0,1)$ (see Fig 2). Express the integral

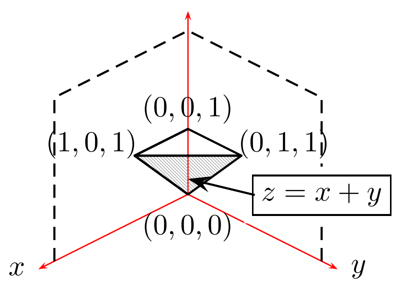

Figure 2: Problem 6

$$\iiint_{\mathcal{E}} f(x, y, z) \ dV$$

as an iterated integral in the order

(a) $dz \ dy \ dx$

$$\int_0^1 \int_0^{1-x} \int_{x+y}^1 f(x,y,z) \ dz \ dy \ dx$$

(b) $dy \ dz \ dx$

$$\int_0^1 \int_x^1 \int_0^{z-x} f(x,y,z) \ dy \ dz \ dx$$

Problem 7. Calculate the surface integral (flux integral)

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\ dS$$

of the vector field

$$\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$$

over the surface $\mathcal{S}$ given by $z = 4 - x^2 - y^2$, $z \geq 0$.

The surface is the graph of $z = f(x, y)$ where $f(x, y) = 4 - x^2 - y^2$ over the domain $D : x^2 + y^2 \leq 4$. Using the *upward* orientation we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot \langle -f_x, -f_y, 1 \rangle\ dA \\
&= \int_D \left\langle -x, -y, 4 - x^2 - y^2 \right\rangle \cdot \langle 2x, 2y, 1 \rangle\ dA \\
&= \iint_D -3x^2 - 3y^2 + 4\ dA \\
&= \int_0^{2\pi} \int_0^2 (4 - 3r^2)\ r\ dr\ d\theta \\
&= 2\pi(2r^2 - \frac{3}{4}r^4)\Big|_0^2 \\
&= -8\pi.
\end{aligned}$$

Problem 8. Calculate the surface integral

$$\iint_{\mathcal{S}} x^2 z^2\ dS$$

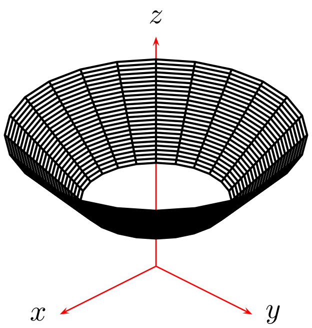

Figure 3: Problem 8

where $\mathcal{S}$ is the part of the cone $z^2 = x^2 + y^2$ between the planes $z = 1$ and $z = 2$ (see Fig 3).

Let us set up the integral in Euclidean coordinates then convert to polar. The surface is the graph of $z = f(x, y)$ where $f(x, y) = \sqrt{x^2 + y^2}$ over the domain $D : 1 \leq x^2 + y^2 \leq 4$. Thus we have

$$\begin{aligned}
\iint_S x^2 z^2 \, dS &= \iint_D x^2(x^2+y^2)\sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA \\
&= \iint_D x^2(x^2+y^2)\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1}\, dA \\
&= \iint_D x^2(x^2+y^2)\sqrt{\frac{x^2+y^2}{x^2+y^2} + 1}\, dA \\
&= \iint_D x^2(x^2+y^2)\ \sqrt{2}\, dA \\
&= \sqrt{2}\int_0^{2\pi}\int_1^2 r^2\cos^2\theta (r^2)\ r\ dr\ d\theta \\
&= \sqrt{2}\int_0^{2\pi}\int_1^2 r^5\cos^2\theta\ dr\ d\theta \\
&= \sqrt{2}\int_0^{2\pi}\int_1^2 r^5\frac{1}{2}(\cos(2\theta)+1)\ dr\ d\theta \\
&= \sqrt{2}\left(\frac{r^6}{6}\Big|_1^2\right)\left(\frac{1}{2}(\theta + \frac{\sin(2\theta)}{2})\Big|_0^{2\pi}\right) \\
&= \sqrt{2}\left(\frac{2^6-1}{6}\right)\left(\frac{(2\pi-0)+\frac{1}{2}(0-0)}{2}\right) \\
&= \sqrt{2}\frac{63\pi}{6} \\
&= \sqrt{2}\frac{21\pi}{2}
\end{aligned}$$

Problem 9. **Use the Divergence Theorem to calculate the flux of the vector field $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ over the surface of the prism in the first quadrant bounded by the coordinate planes, the plane $x + y = 1$, and the plane $z = 1$ (see Fig 4), with outward orientation.**

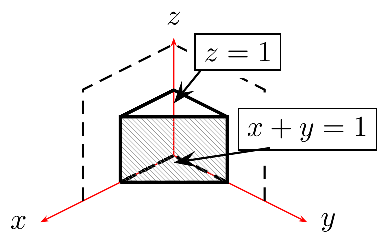

Figure 4: Problem 9

Since $\mathcal{S}$ is a closed surface with outward orientation, we have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_{\mathcal{E}} \operatorname{div}(\mathbf{F})\ dV \\
&= \iiint_{\mathcal{E}} y + z + x\ dV \\
&= \int_0^1 \int_0^{1-x} \int_0^1 y + z + x\ dz\ dy\ dx \\
&= \int_0^1 \int_0^{1-x} y + x + \frac{1}{2}\ dy\ dx \\
&= \int_0^1 \frac{1}{2}(1-x)^2 + x(1-x) + \frac{1-x}{2}\ dx \\
&= \int_0^1 1 - \frac{x}{2} - \frac{x^2}{2}\ dx \\
&= x - \frac{x^2}{4} - \frac{x^3}{6} \\
&= \frac{7}{12}
\end{aligned}$$