Solutions for Math 13 Exam 2, Fall 2009

November 23, 2009

1. The volume of such a box is $V(x, y, z) = xyz$, so we look to maximize $V(x, y, z)$ subject to the constraint that $g(x, y, z) = x + 2y + 3z = 6$. For the Lagrange multipliers, we then have the equations

$$\nabla V(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = x + 2y + 3z = 6.$$

Taking the gradients, this becomes

$$yz = \lambda \quad (1)$$
$$xz = 2\lambda \quad (2)$$
$$xy = 3\lambda \quad (3)$$
$$x + 2y + 3z = 6. \quad (4)$$

Note that if $\lambda = 0$, then at least two of x, y, and z must be zero, which would make $V(x, y, z) = 0$, which does not maximize the volume. So, $\lambda \neq 0$. Then, taking the ratio of Equations (2) and (1) gives that $\frac{x}{y} = 2$, or $2y = x$. Similarly, taking the ratio of Equations (3) and (1) gives that $\frac{x}{z} = 3$, or $3z = x$. Substituting these into Equation (4) gives $x + x + x = 6$, or $x = 2$. This leads to $y = 1$ and $z = \frac{2}{3}$ (and $\lambda = \frac{2}{3}$). The maximum volume is, then, $V(2, 1, \frac{2}{3}) = \frac{4}{3}$.

2. First convert the equations into spherical coordinates. The sphere of radius 3 ($x^2 + y^2 + z^2 = 9$) becomes $\rho = 3$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$. The equation for the cone, $z = \sqrt{x^2 + y^2}$, yields $\tan(\phi) = 1$, or $\phi = \frac{\pi}{4}$, with $0 \le \rho$ and $0 \le \theta \le 2\pi$. Finally, the xy-plane is the surface with $\phi = \frac{\pi}{2}$, $0 \le \rho$, and $0 \le \theta \le 2\pi$.

   Taking the intersections of these surfaces, we find that the volume described is that with $0 \le \rho \le 3$, $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$, and $0 \le \theta \le 2\pi$. This means that the volume is

$$V = \iiint_E 1 dV = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^3 \rho^2 \sin(\phi) d\rho d\phi d\theta$$
$$= \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left. \frac{\rho^3}{3} \right|_{\rho=0}^{\rho=3} \sin(\phi) d\phi d\theta = 9 \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(\phi) d\phi d\theta$$
$$= 9 \int_0^{2\pi} -\cos(\phi)|_{\phi=\frac{\pi}{4}}^{\phi=\frac{\pi}{2}} d\theta = \frac{9\sqrt{2}}{2} \int_0^{2\pi} d\theta = 9\sqrt{2}\pi.$$

3. (a) Fixing x and y, in any z-slice, we have $0 \le z \le 1 - y^2$. Fixing x, then the y-coordinate runs from x to 1. Finally, $0 \le x \le 1$. This gives

$$\iint_E f(x,y,z) dV = \int_0^1 \int_x^1 \int_0^{1-y^2} f(x,y,z) dz dy dx.$$

   (b) Fixing y and z, in any x-slice, we have $0 \le x \le y$. Then fixing y, we again have $0 \le z \le 1 - y^2$. Finally, $0 \le y \le 1$. This gives

$$\iint_E f(x,y,z) dV = \int_0^1 \int_0^{1-y^2} \int_0^y f(x,y,z) dx dz dy.$$

4. In cylindrical coordinates, we have that $x^2+y^2=r^2$, but do not change z. The region $-3 \le x \le 3$, $0 \le y \le \sqrt{9-x^2}$ describes the upper semicircle of radius 3 in the xy-plane. In polar coordinates, this becomes $0 \le r \le 3$, $0 \le \theta \le \pi$. The bounds in z become, $\pm\sqrt{9-r^2}$, giving

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}}(x^2+y^2)^{\frac{3}{2}}dzdydx = \int_0^3\int_0^\pi\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r^3\cdot rdzd\theta dr = \int_0^3\int_0^\pi\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r^4 dzd\theta dr.$$

5. Note that C is the positively-oriented simple closed contour that bounds the upper half of the unit circle, D. So, we can apply Green's Theorem to help us evaluate this integral. Writing $\oint_C y^3dx - x^3dy = \oint_C P(x,y)dx + Q(x,y)dy$ for $P(x,y)=y^3$ and $Q(x,y)=-x^3$, Green's Theorem gives

$$\oint_C y^3dx - x^3dy = \oint_C P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \left(-3x^2-3y^2\right) dA.$$

Changing to polar coordinates, we get

$$\oint_C y^3dx - x^3dy = -3\int_0^\pi\int_0^1 r^2\cdot rdrd\theta = -3\int_0^\pi\int_0^1 r^3drd\theta = -3\int_0^\pi \left.\frac{r^4}{4}\right|_{r=0}^{r=1} d\theta = -\frac{3\pi}{4}.$$

Alternately, we could parametrize C as two curves, C_1 being the upper half circle and C_2 as the segment along the x-axis from -1 to +1.

Parametrizing C_1 by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ for $0 \le t \le \pi$, we get $x(t)=\cos(t)$, $x'(t) = -\sin(t)$, $y(t) = \sin(t)$, and $y'(t)=\cos(t)$. Plugging this in gives us

$$\begin{aligned}
\int_{C_1} y^3dx - x^3dy &= \int_0^\pi (-\sin^4(t) - \cos^4(t))dt \\
&= -\int_0^\pi \left(\sin^2(t)\cdot(1-\cos^2(t)) + \cos^2(t)\cdot(1-\sin^2(t))\right)dt = -\int_0^\pi \left(1-2\sin^2(t)\cos^2(t)\right)dt \\
&= -\int_0^\pi \left(1-\frac{1}{2}\sin^2(2t)\right)dt = -\int_0^\pi\left(\frac{3}{4}+\frac{1}{4}\left(1-2\sin^2(2t)\right)\right)dt \\
&= -\int_0^\pi\left(\frac{3}{4}+\frac{1}{4}\cos(4t)\right)dt = -\left(\frac{3t}{4}+\frac{\sin(4t)}{16}\right)\Bigg|_{t=0}^{t=\pi} = -\frac{3\pi}{4}.
\end{aligned}$$

Parametrizing C_2 by $\mathbf{r}(t) = t\mathbf{i} + 0\mathbf{j}$ for $-1 \le t \le 1$, we get $x(t) = t$, $x'(t) = 1$, $y(t) = 0$, and $y'(t) = 0$. Plugging this in gives us

$$\int_{C_2} y^3dx - x^3dy = \int_{-1}^1 (0\cdot 1 - t^3\cdot 0)dt = 0.$$

Thus, $\oint_C y^3dx - x^3dy = \int_{C_1} y^3dx - x^3dy + \int_{C+2} y^3dx - x^3dy = -\frac{3\pi}{4}$.

6. (a) Given $\mathbf{r}(u,v) = (u^2+v^2)\mathbf{i} + uv\mathbf{j} + u\mathbf{k}$, we can find two vectors tangent to the surface at $(5,2,1)$. First compute

$$\begin{aligned}
\mathbf{r}_u(u,v) &= 2u\mathbf{i} + v\mathbf{j} + \mathbf{k} \\
\mathbf{r}_v(u,v) &= 2v\mathbf{i} + u\mathbf{j} + 0\mathbf{k},
\end{aligned}$$

and then evaluate these at $(u,v) = (1,2)$,

$$\begin{aligned}
\mathbf{r}_u(1,2) &= 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \\
\mathbf{r}_v(1,2) &= 4\mathbf{i} + 1\mathbf{j} + 0\mathbf{k}.
\end{aligned}$$

The normal vector to S at P must be orthogonal to both of these vectors, so

$$\mathbf{n} = \mathbf{r}_u(1,2)\times\mathbf{r}_v(1,2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 4 & 1 & 0 \end{vmatrix} = (2\cdot 0 - 1\cdot 1)\mathbf{i} - (2\cdot 0 - 1\cdot 4)\mathbf{j} + (2\cdot 1 - 2\cdot 4)\mathbf{k} = -\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}.$$

(b) The tangent plane to S at P is the plane through P whose normal vector is $\mathbf{n} = -\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$. So, the equation for the plane is

$$\mathbf{n} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{n} \cdot (5\mathbf{i} + 2\mathbf{j} + 1\mathbf{k})$$
$$-x + 4y - 6z = -3.$$

7. (a) First, we parameterize the curve, C, as $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, for $0 \le t \le \frac{\pi}{2}$. Then, $\mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$, and $|\mathbf{r}'(t)| = 1$. So,

$$\int_C x^4 y ds = \int_0^{\frac{\pi}{2}} \cos^4(t)\sin(t)|vecr'(t)|dt = \int_0^{\frac{\pi}{2}} \cos^4(t)\sin(t)dt = -\frac{\cos^5(t)}{5}\Big|_{t=0}^{t=\frac{\pi}{2}} = \frac{1}{5}.$$

(b) Along this curve, $\mathbf{F}(x(t), y(t)) = e^{\ln t}\mathbf{i} + e^{\ln 2t}\mathbf{j} = t\mathbf{i} + 2t\mathbf{j}$. Also, for $\mathbf{r}(t) = \ln(t)\mathbf{i} + \ln(2t)\mathbf{j}$, $\mathbf{r}'(t) = \frac{1}{t}\mathbf{i} + \frac{1}{t}\mathbf{j}$. So,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^4 \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)dt = \int_1^4 (1+2)dt = 9.$$

Alternately, we could notice that the vector field, $\mathbf{F}(x, y) = e^x\mathbf{i} + e^y\mathbf{j}$, satisfies $\frac{\partial}{\partial x}e^y = \frac{\partial}{\partial y}e^x$, so the field is conservative.

To find a potential function, $f(x, y)$ notice that $\frac{\partial f}{\partial x} = e^x$, so $f(x, y) = e^x + K(y)$. Similarly, $\frac{\partial f}{\partial y} = e^y$, so $K(y) = e^y + k$, and $f(x, y) = e^x + e^y + k$.

Now, we can apply the Fundamental Theorem for Line Integrals, and

$$\int_c \mathbf{F} \cdot d\mathbf{r} = f(\ln(4), \ln(8)) - f(\ln(1), \ln(2)) = e^{\ln(4)} + e^{\ln(8)} + k - \left(e^{\ln(1)} + e^{\ln(2)} + k\right) = 9.$$

8. (a) If $\mathbf{F} = \nabla f$, then $\frac{\partial f}{\partial x} = \cos x + 2yz$. Integrating with respect to x, this gives

$$f(x, y, z) = \sin x + 2xyz + K(y, z),$$

where $K(y, z)$ is an unknown function that is independent of x. Computing $\frac{\partial f}{\partial y} = 2xz + \frac{\partial K}{\partial y}$, we see that $\frac{\partial K}{\partial y} = 0$ and, so $K(y, z) = K(z)$. Next computing $\frac{\partial f}{\partial z} = 2xy + \frac{\partial K}{\partial z}$, we get $\frac{\partial K}{\partial z} = z$, giving $K(z) = \frac{1}{2}z^2 + k$. This gives

$$f(x, y, z) = \sin x + 2xyz + \frac{1}{2}z^2 + k.$$

(b) By the fundamental theorem for line integrals,

$$\int_C \mathbf{F} \cdot \mathbf{r} = f(\pi, \pi, \pi) - f(0, 0, 0) = 2\pi^3 + \frac{\pi^2}{2}.$$