Math 13, Exam 1 Solutions

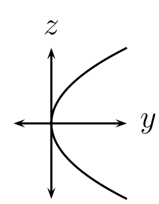
1. (a) False – the dot product of two vectors is a scalar. In this case, $\vec{i} \cdot \vec{j} = 0$.
   (b) False – this is a hyperboloid of two sheets.
   (c) True – a normal direction for the plane is $\langle 0, 1, 0 \rangle$ and a direction vector for the line is also $\langle 0, 1, 0 \rangle$.
   (d) False. They are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.
   (e) False.

2. (a) A normal direction for the plane is $\langle 2, -1, 2 \rangle \times \langle 3, 1, 1 \rangle = \langle -3, 4, 5 \rangle$. Using the point $(1, 2, 3)$ gives the equation

$$-3(x-1) + 4(y-2) + 5(z-3) = 0\,.$$

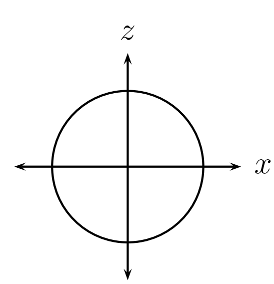
   (b) A normal direction for the plane is $\langle 2, -3, 1 \rangle$, so we get the parametric equations

$$\begin{aligned} x(t) &= 1 + 2t \\ y(t) &= 2 - 3t \\ z(t) &= 6 + t\,. \end{aligned}$$



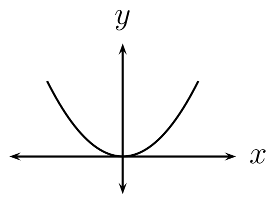
3. (a)

Figure 1: $x = 0 : y = z^2$



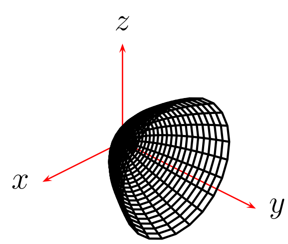
(b)

Figure 2: $y = 1 : x^2 + z^2 = 1$



(c)

Figure 3: $z = 0 : y = x^2$



(d)

Figure 4: $y = x^2 + z^2$

4. (a) We have

$$\vec{r}'(t) = \langle -3\sin 3t, 3\cos 3t, 4\rangle\,,$$

so $\vec{r}'(\frac{\pi}{4}) = \langle -\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}, 4\rangle$. Also, $\vec{r}(-\frac{3\sqrt{2}}{2}) = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \pi\rangle$. Thus parametric equations of this line are

$$\begin{aligned} x(t) &= -\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}t \\ y(t) &= \frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}t \\ z(t) &= \pi + 4t\,. \end{aligned}$$

(b) $|\vec{r}'(t)| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 16} = 5$. Thus the distance travelled from time $t = 0$ to $t = 2$ is

$$\int_0^2 |\vec{r}'(t)|dt = 5\int_0^2 dt = 10\,.$$

5. (a) $f_x = \dfrac{x}{\sqrt{x^2+4y^2}}$ and $f_y = \dfrac{4y}{\sqrt{x^2+4y^2}}$, so $f_x(3,2) = \frac{3}{5}$ and $f_y(3,2) = \frac{8}{5}$.

Thus

$$\begin{aligned} L(x, y) &= f_x(3,2)(x-3) + f_y(3,2)(y-2) + f(3,2) \\ &= \frac{3}{5}(x-3) + \frac{8}{5}(y-2) + 5 \\ &= \frac{3}{5}x + \frac{8}{5}y\,. \end{aligned}$$

(b) $L(3.1, 1.9) = \frac{3}{5}(3.1) + \frac{8}{5}(1.9)$.

6. Set $F = x^2y + yz^2 - e^{xz}$; then S is the level surface $F = 3$, and a normal direction to the tangent plane to S at $(2,1,0)$ is given by $\nabla F(2,1,0)$. Now, $\nabla F = \langle 2xy - ze^{xz},\, x^2 + z^2,\, 2yz - xe^{xz}\rangle$, so $\nabla F(2,1,0) = \langle 4, 4, -2\rangle$ and an equation of the tangent plane is

$$4(x-2) + 4(y-1) - 2z = 0\,.$$

7. (a) f increases fastest in the direction of ∇f. Now, $\nabla f = \langle 6x, 8y\rangle$, so $\nabla f(1,1) = \langle 6, 8\rangle$. We want the unit vector in this direction. Since $|\nabla f(1,1)| = 10$, the unit vector in the direction of $\nabla f(1,1)$ is $\frac{1}{10}\nabla f(1,1) = \langle \frac{3}{5}, \frac{4}{5}\rangle$.

   (b) $|\vec{v}| = 5$, so the unit vector $\vec{u}$ in the direction of $\vec{v}$ is $\frac{1}{5}\vec{v} = \langle \frac{4}{5}, \frac{3}{5}\rangle$. Thus

$$D_{\vec{u}}f(1,1) \;=\; \vec{u}\cdot\nabla f(1,1) \;=\; \left\langle \frac{4}{5}, \frac{3}{5}\right\rangle \cdot \langle 6, 8\rangle \;=\; \frac{48}{5}\,.$$

8. We have $f_x = 3x^2 - 3y$ and $f_y = -3x + y$. Thus $f_{xx} = 6x$, $f_{xy} = -3$, and $f_{yy} = 1$, so

$$D \;=\; f_{xx}f_{yy} - (f_{xy})^2 \;=\; 6x - 9\,.$$

- $D(0,0) = -9 < 0$, so $(0,0)$ is a saddle point of f.
- $D(3,9) = 9 > 0$ and $f_xx(3,9) = 18 > 0$, so $(3,9)$ is a local minimum of f.

9. We'll break this up into three steps.

   (i) We first find and evaluate the critical points of f on the interior of D. Now, $f_x = x^2$ and $f_y = 2y$. Solving the system $x^2 = 0$ and $2y = 0$ gives the single critical point $(0,0)$, which is indeed inside D; and $f(0,0) = 0$.

   (ii) We now find absolute extrema of f on the boundary $x^2 + y^2 = 1$ of D. Plugging $y^2 = 1 - x^2$ into f, we obtain the function $g(x) = \frac{1}{3}x^3 - x^2 + 1$ on the interval $[-1, 1]$. Now, $g' = x^2 - 2x$; solving $g' = 0$ gives $x^2 - 2x = 0$ and hence g has critical points at $x = 0, 2$. But 2 isn't in our interval, so we only consider $x = 0$, which gives $g(0) = 1$. We also need to check the endpoints: $g(-1) = -\frac{1}{3}$ and $g(1) = \frac{1}{3}$.

   (iii) Comparing (i) and (ii), we see that $-\frac{1}{3}$ is the absolute minimum value and 1 is the absolute maximum value of f on D.