Math 13 - Final Exam
- Solutions -

May 12, 2008

1. $\vec{n} = \langle 1, 3, -7 \rangle$, $P = (-13, 0, 1)$

a) $1(x+13) + 3(y-0) - 7(z-1) = 0$

or $\boxed{x + 3y - 7z + 20 = 0}$

b) Largest directional derivative occurs in direction of $\vec{\nabla} f$.

$\Rightarrow \vec{\nabla} f \parallel \vec{u} \Rightarrow \vec{\nabla} f = \lambda \vec{u}$

So: $\langle 3, f_y(1,0,5), f_z(1,0,5) \rangle = \frac{\lambda}{3} \langle 1, 2, 2 \rangle$

$\Rightarrow \frac{\lambda}{3} = 3 \Rightarrow \boxed{\lambda = 9} \Rightarrow \boxed{f_y(1,0,5) = f_z(1,0,5) = 6}$

2. $f(x,y) = xy + x^2 + y^2$, $D = [x^2 + y^2 \leq 1]$

$$\left.\begin{aligned} f_x &= y + 2x = 0 \Rightarrow y = -2x \\ f_y &= x + 2y = 0 \Rightarrow x = -2y \end{aligned}\right\} \Rightarrow \begin{aligned} &x = -2(-2x) \\ &4x = x \Leftrightarrow 3x = 0 \\ &\Rightarrow x = 0 \end{aligned}$$

So: $CP = \{(0,0)\}$ and $(0,0) \in D$ ✓

$\boxed{f(0,0) = 0}$

To find extreme values of f on ∂D, use Lagrange multipliers.

$f(x,y) = xy + x^2 + y^2$ subject to $\underbrace{x^2 + y^2}_{g(x,y)} = 1$

$$\left\{\begin{array}{l} y+2x=\lambda(2x) \\ x+2y=\lambda(2y) \\ x^2+y^2=1 \end{array}\right. \Rightarrow \begin{array}{l}(x+y)+2(x+y)=2\lambda(x+y) \\ 3(x+y)=2\lambda(x+y) \\ (x+y)(2\lambda-3)=0\end{array}$$

Case 1: $x=-y \Rightarrow 2x^2=1 \Rightarrow x=\pm\frac{1}{\sqrt{2}},\ y=\mp\frac{1}{\sqrt{2}}$

and $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{1}{2}$

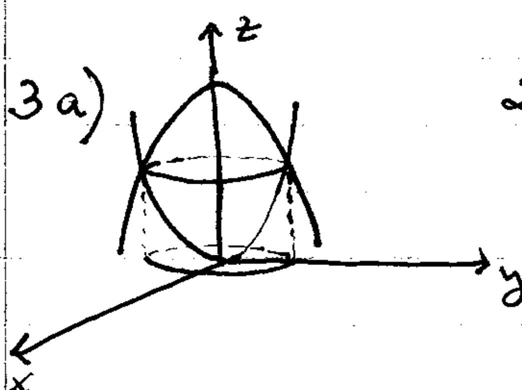
Case 2: $\lambda=\frac{3}{2} \Rightarrow \left.\begin{array}{l} y+2x=3x \\ x+2y=3y\end{array}\right\} \Leftrightarrow x-y=0 \Rightarrow x=y$

So: $x=y=\pm\frac{1}{\sqrt{2}}$

and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$

Hence, Absolute Minimum: $f(0,0)=0$
Absolute Maximum: $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)=\frac{3}{2}$

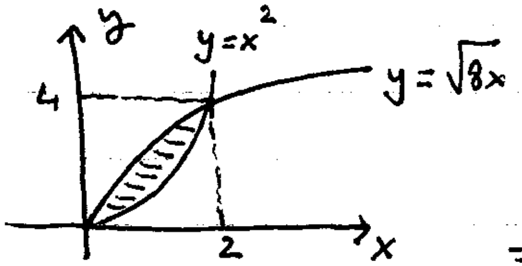
3 a)



$2-(x^2+y^2)=x^2+y^2$
$\Leftrightarrow x^2+y^2=1$ (intersection between paraboloids)

$$V=\iiint_E 1\,dV=\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r\,dz\,dr\,d\theta$$

b)



$$\int_0^2\int_{x^2}^{\sqrt{8x}} x^2y\,dy\,dx =$$

$$=\int_0^4\int_{y^2/8}^{\sqrt{y}} x^2y\,dx\,dy$$

4. $\vec{a}(t) = \langle e^{-t}, 0, 1 \rangle$, $\vec{r}(0) = \langle 0, 0, 300 \rangle$
$\vec{v}(0) = \langle 0, 50, 0 \rangle$

$\vec{v}(t) = \langle -e^{-t}, 0, t \rangle + \vec{C}_1$
$\vec{v}(0) = \langle -1, 0, 0 \rangle + \vec{C}_1 = \langle 0, 50, 0 \rangle \Rightarrow \vec{C}_1 = \langle 1, 50, 0 \rangle$

$$\Rightarrow \vec{v}(t) = \langle 1 - e^{-t}, 50, t \rangle$$

$\vec{r}(t) = \langle t + e^{-t}, 50t, \frac{1}{2}t^2 \rangle + \vec{C}_2$

$\vec{r}(0) = \langle 1, 0, 0 \rangle + \vec{C}_2 = \langle 0, 0, 300 \rangle \Rightarrow \vec{C}_2 = \langle -1, 0, 300 \rangle$

$\Rightarrow \vec{r}(t) = \langle -1 + t + e^{-t}, 50t, 300 + \frac{1}{2}t^2 \rangle$

and so $\boxed{r(10) = \langle 9 + e^{-10}, 500, 350 \rangle}$

5. $\vec{F} = \langle yz, xz + 3y, xy + 5 \rangle$

a) $$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ yz & xz + 3y & xy + 5 \end{vmatrix} = \langle 0, 0, 0 \rangle,$$ so $\vec{F}$ conservative

b) $f_x = yz \Rightarrow f = \int yz \, dx = xyz + C_1(y, z)$

$f_y = xz + 3y \Rightarrow f = \int (xz + 3y) \, dy = xyz + \frac{3}{2}y^2 + C_2(x, z)$

$f_z = xy + 5 \Rightarrow f = \int (xy + 5) \, dz = xyz + 5z + C_3(x, y)$

$$\Rightarrow \boxed{f(x, y, z) = xyz + \frac{3}{2}y^2 + 5z + K}$$

c) $\int_C \vec{F} \cdot d\vec{r} = 0$, since $\vec{F}$ conservative and C-closed.

6. $\int_C \underbrace{(y + e^{x^2})}_{P} dx + \underbrace{(2x + \ln(y^2))}_{Q} dy =$

$= \iint_D (Q_x - P_y)\, dA = \iint_D (2-1)\, dA = A(D) = \boxed{6}$

7. Sphere and cone intersect when:

$x^2+y^2 + 3(x^2+y^2) = 9 \iff 4(x^2+y^2) = 9$

or $x^2+y^2 = \frac{9}{4}$

$$\boxed{\vec{r}(x,y) = \langle x, y, \sqrt{3(x^2+y^2)} \rangle, \quad (x,y) \in D = [x^2+y^2 \leq \tfrac{9}{4}]}$$

8. $z = x^2 - y^2$ inside $x^2+y^2=4$, $\vec{F} = \langle \overset{P}{x}, \overset{Q}{-y}, \overset{R}{1} \rangle$

Using upward normal,

$\iint_S \vec{F} \cdot d\vec{S} = \iint_{[x^2+y^2 \leq 4]} (-P z_x - Q z_y + R)\, dA =$

$= \iint_{[x^2+y^2 \leq 4]} (-2x^2 - 2y^2 + 1)\, dA = \int_0^{2\pi} \int_0^2 (-2r^2+1)\, r\, dr\, d\theta$

$= 2\pi \left(-\frac{r^4}{2} + \frac{r^2}{2} \right) \Big|_0^2 = 2\pi(-8+2) = \boxed{-12\pi}$

or 12π using downward normal

9. $\vec{F} = \langle x^2 y,\ 2xz,\ yz^3 \rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E (\operatorname{div}\vec{F})\,dV = \iiint_E (2xy + 3yz^2)\,dV$$

$$= \iint_D \left(\int_0^5 (2xy + 3yz^2)\,dz \right) dA$$

$$= \iint_D (2xyz + yz^3)\Big|_{z=0}^{z=5} = \iint_D (10xy + 125y)\,dA$$

$$= \int_0^1 \int_0^1 (10xy + 125y)\,dy\,dx$$

$$= \int_0^1 \left(5xy^2 + \frac{125}{2}y^2\right)\Big|_{y=0}^{y=1} dx = \int_0^1 \left(5x + \frac{125}{2}\right) dx$$

$$= \left(\frac{5}{2}x^2 + \frac{125}{2}x\right)\Big|_0^1 = \frac{130}{2} = \boxed{65}$$

10. $\overrightarrow{\operatorname{curl}}\,\vec{F} = \langle 0, -2x, 0 \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \overrightarrow{\operatorname{curl}}\,\vec{F}\cdot d\vec{S} = \iint_D \langle 0, -2x, 0\rangle \cdot \langle 1, 2, 1\rangle\,dA$$

$$= \iint_D -4x\,dA = -4\int_0^2 \int_0^{1-\frac{1}{2}x} x\,dy\,dx = \boxed{-\frac{8}{3}}$$