The Catenary curve

Question: what is the shape of the St. Louis Gateway Arch?

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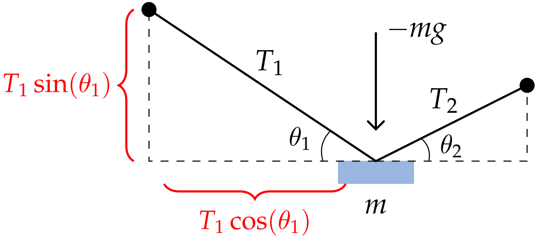
Not a parabola!

New Question: find the shape of a hanging chain.

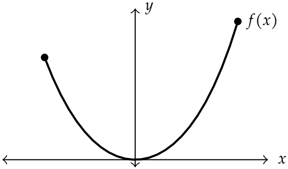
1. A mass hanging from two chains:



Balance forces!

- x-direction: $T_1 \cos(\theta_1) = T_2 \cos(\theta_2)$.
- y-direction: $T_1 \sin(\theta_1) + T_2 \sin(\theta_2) = mg$.

2. Hanging a chain: the set-up

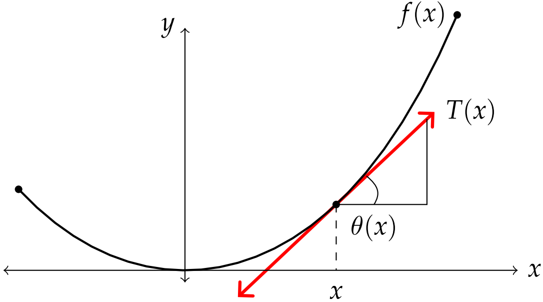


Coordinate system: $f(0) = 0,\ f'(0) = 0.$

- $\rho =$ density of chain
- $A =$ cross-sectional areas
- $$s(x) = \text{arclength of } f \text{ from } 0 \text{ to } x = \int_0^x \sqrt{1 + (f'(t))}\, dt$$

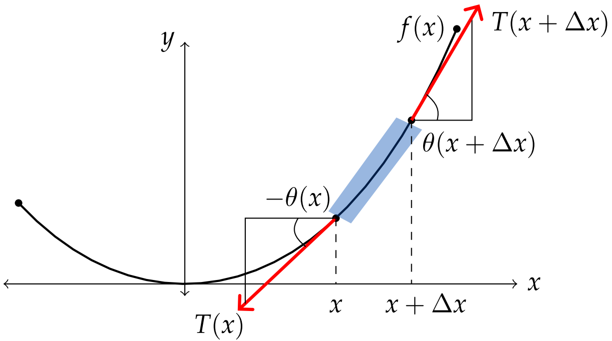
Play the balancing forces game for the hanging chain!

Let $\Delta x > 0$. Balance the forces for a mass at position x with horizontal length Δx:



Play the balancing forces game for the hanging chain!

Let $\Delta x > 0$. Balance the forces for a mass at position x with horizontal length Δx:



x-direction: balance horizonal forces

$$T(x)\cos(\theta(x)) = T(x+\Delta x)\cos(\theta(x+\Delta x)) \qquad \text{for all } x!$$

$$\implies T(x)\cos(\theta(x)) = B \qquad \text{constant}$$

y-direction: balance vertical forces

$$\begin{aligned} T(x+\Delta x)\sin(\theta(x+\Delta x)) - T(x)\sin(\theta(x)) \\ &= mg \\ &= \rho A g(\text{arclength } x \text{ to } x+\Delta x) \\ &= \rho A g\,(s(x+\Delta x) - s(x)) \end{aligned}$$

Set equal and divide by Δx:

$$\frac{T(x+\Delta x)\sin(\theta(x+\Delta x)) - T(x)\sin(\theta(x))}{\Delta x} = \frac{\rho A g\,(s(x+\Delta x) - s(x))}{\Delta x}$$

$$\Delta x \to 0 \Longrightarrow \frac{d}{dx} T(x)\sin(\theta(x)) = \rho A g \frac{d}{dx} s(x)$$

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Recall: $T(x)\cos(\theta(x)) = B \implies T(x) = \dfrac{B}{\cos(\theta(x))}$. Substitute:

$$\begin{aligned}
B\frac{d}{dx}\frac{\sin(\theta(x))}{\cos(\theta(x))} &= \rho A g \frac{d}{dx}s(x) \\
\frac{d}{dx}\tan(\theta(x)) &= \frac{\rho A g}{B}\frac{d}{dx}s(x) \\
\frac{d}{dx}f'(x) &= \frac{\rho A g}{B}\frac{d}{dx}s(x) \\
f''(x) &= K\frac{d}{dx}\int_0^x \sqrt{1 + (f'(t))^2}\,dt \\
f''(x) &= K\sqrt{1 + (f'(x))^2}
\end{aligned}$$

Solve the second order, nonlinear, separable differential equation to get the curve that models a hanging chain:

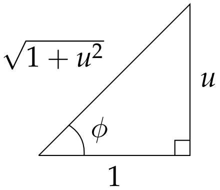
$$f''(x) = K\sqrt{1 + (f'(x))^2}$$

Step 1. Let $u(x) = f'(x)$, so $u'(x) = f''(x)$.

Exercise. Replace f' with u and solve to find a relationship between u and x.

$$
\begin{aligned}
u'(x) &= K\sqrt{1+(u(x))^2} \\
\int \frac{du}{\sqrt{1+u^2}} &= \int K\,dx \\
\int \frac{\sec^2(\phi)\,d\phi}{\sqrt{1+\tan^2(\theta)}} &= Kx + C && u = \tan(\phi),\ du = \sec^2(\phi) \\
\int \sec(\phi)\,d\phi &= Kx + C \\
\ln|\sec(\phi)+\tan(\phi)| &= Kx + C \\
\ln|\sqrt{1+u^2}+u| &= Kx + C && \text{right triangle} \\
\sqrt{1+u^2}+u &= Ce^{Kx} && C > 0
\end{aligned}
$$

How to write u as a function of x?



$$u = \frac{u}{1} = \tan(\phi) = \frac{\text{opp}}{\text{adj}}$$

$$\sec(\phi) = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{1+u^2}}{1}$$

Write u as a function of x:

$$u + \sqrt{1+u^2} = Ce^{Kx}$$
$$Ce^{Kx}(u - \sqrt{1+u^2}) = (u + \sqrt{1+u^2})(u - \sqrt{1+u^2})$$
$$Ce^{Kx}(u - \sqrt{1+u^2}) = u^2 - (1+u^2) = -1$$
$$\implies u - \sqrt{1+u^2} = -Ce^{-Kx}$$

Putting these together:

$$u + \sqrt{1+u^2} + u - \sqrt{1+u^2} = Ce^{Kx} - Ce^{-Kx}$$
$$2u = Ce^{Kx} - Ce^{-Kx}$$
$$u = \frac{C(e^{Kx} - e^{-Kx})}{2}$$

What is C? We have an initial condition!

$$u(0) = f'(0) = 0$$
$$\implies 0 + \sqrt{1+0} = Ce^0 \implies C = 1$$

Step 2. Now,

$$u = f'(x) = \frac{e^{Kx} - e^{-Kx}}{2}$$

is another separable differential equation! Solve by integrating:

$$f(x) = \int f'(x)dx = \frac{1}{2}\int e^{Kx} - e^{-Kx}\,dx$$
$$= \frac{1}{2K}(e^{Kx} + e^{-Kx}) + C_1$$

Solve for C_1:

$$0 = f(0) = \frac{1}{2K}(1+1) + C_1$$
$$0 = \frac{1}{K} + C_1$$
$$C_1 = -\frac{1}{K}$$

Recall: $K = \frac{\rho A g}{B}$. The final solution is:

$$f(x) = \frac{B\left(e^{\frac{\rho A g}{B}x} + e^{-\frac{\rho A g}{B}x}\right)}{2\rho A g} - \frac{B}{\rho A g}$$

The curve is a hyperbolic cosine function!

- $2\cosh(x) = e^x + e^{-x}$
- $2\sinh(x) = e^x - e^{-x}$

$$\frac{d^2}{dx^2}\cosh(x) = \frac{d}{dx}\sinh(x) = \cosh(x)$$

How to build the St. Louis arch?
A hanging chain is perfectly balanced with respect to tension and gravity: flip it over!

630 ft

630 ft

$$f(x) = -630\cosh\left(\frac{x}{239.2}\right) + 1260$$

defined on the domain $|x| \leq 315$. This is a <u>catenary curve</u>!