1. The square root of -1

We begin by introducing a symbol i to represent a new kind of number that has the property that $i^2 = -1$; we'll use the term *complex number* to mean any expression of the form $a + bi$ where a and b are real. For a complex number $z = a + bi$, we'll call a its *real part* and we'll call b its *imaginary part*. In particular, two complex numbers $a + bi$ and $c + di$ are equal if and only if their real parts and imaginary parts match (that is, $a = c$ and $b = d$).

We'll declare that the laws of arithmetic apply to our new numbers in the following form:

$$k(a + bi) = ka + kbi \ , \qquad (a + bi) + (c + di) = (a + c) + (b + d)i \ ,$$

which makes all the other arithmetic laws like commutativity and distributivity work just like usual. Also, if $z = a + bi$, then we will define its *complex conjugate* as $\bar{z} = a - bi$.

That means that the equation $x^2 + 1 = 0$, which has no solutions when x is a real number, now has two solutions, $\pm i$ (which are complex conjugates of each other).

Example 1. $z = 3 + 4i$ is 3 has real part 3, the imaginary part 4, and complex conjugate $3 - 4i$. The real part of i is 0, the imaginary part is 1, and the complex conjugate is $-i$.

2. Arithmetic of complex numbers

It's really easy to add and subtract complex numbers just by adding and subtracting their real and imaginary parts. How about multiplication and division?

The multiplication formula is a little funny-looking, but it's very easy to work out by using the so-called foil rules as usual:

$$(a + bi)(c + di) = ac + adi + bci + bd(i^2) = ac + (ad + bc)i + (-1)bd = (ac - bd) + (ad + bc)i.$$

How about division? Here, the thing to notice is that you can make denominators real by the same process you used in high school to "rationalize" denominators, by multiplying by the complex conjugate of the denominator.

$$\frac{a + bi}{c + di} = \frac{(a + bi)}{(c + di)} \frac{(c - di)}{(c - di)} = \frac{(a + bi)(c - di)}{c^2 + d^2} \ .$$

Example 2.

$$\frac{3 + i}{i} = \frac{(3 + i)}{(i)} \frac{(-i)}{(-i)} = \frac{1 - 3i}{1} = 1 - 3i.$$

3. Euler's formula and De Moivre's Theorem

Leonhard Euler—the same guy who figured out that $\sum \frac{1}{k^2} = \frac{\pi^2}{6}$ back in 1735—had many ingenious ideas about how to use i in the calculus of infinite series.

We know that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^k}{k!} + \cdots$$

And we've already seen that by plugging in various real numbers like $x = 2$ or $x = 10$, we can get power series expressions that add up to values like e^2, e^{10}, etc.

How about plugging in an imaginary value like $x = i\theta$? Well, we obtain

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots = 1 + i\theta + \frac{-\theta^2}{2} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

and the numerators will cycle between positive real, positive imaginary, negative real, negative imaginary, and so on repeating.

If we separate out the real and imaginary parts, we get

$$e^{i\theta} = \left[1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}\right] + i \cdot \left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right]$$

and lo and behold, with our knowledge of basic power series, this can be rewritten in the remarkable formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta.}$$

This is called **Euler's formula**, and it works for any real number θ. (Note: a word should be said about why this is valid because $i\theta$ is not in the usual interval of convergence for e^x. It's legitimate because the right-hand side has convergent series for its real and imaginary parts, so it represents a unique complex number.)

Now we can state **De Moivre's Theorem**. When we write it in its most compact form, it will look obvious!

$$\left[e^{i\theta}\right]^n = e^{in\theta}.$$

But to make it look more impressive, notice that this can be rewritten as follows:

$$[\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta),$$

which doesn't look obvious at all.

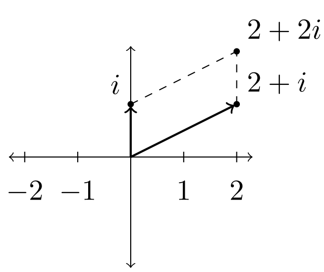
Example 3. *Give formulas for* $\sin(2\theta)$ *and* $\cos(2\theta)$. De Moivre tells us that $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$. Squaring the expression on the left, we get $(\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta$, and so setting real and imaginary parts equal, we obtain

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta\ , \qquad \sin(2\theta) = 2\sin\theta\cos\theta.$$

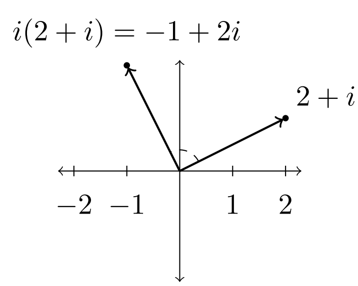
We've proved the double-angle formulas!

4. Rectangular versus polar coordinates

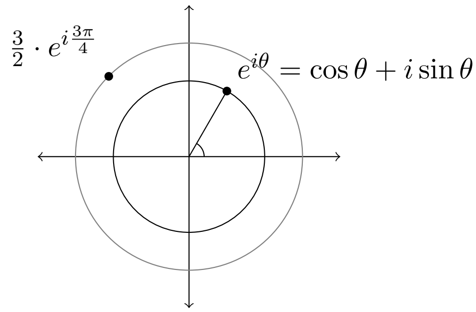
A key idea to help complex numbers seem less strange (and "imaginary") is to plot the numbers on a plane. We will use the y-axis as the imaginary axis and the x-axis as the real axis, so that the point $a + bi$ will be plotted in position (a, b) in the plane. Then it is true (and you can check) that *addition of complex numbers corresponds to vector addition in the plane.* Multiplication by $a + bi$ is a bit harder to picture, but one observation that is helpful is that *multiplication by* i *is rotation by* $\pi/2$ *about the origin.*



Adding i



Multiplying by i



Representing $re^{i\theta}$

Finally, we can visualize Euler's formula: $e^{i\theta}$ is a parametric expression for the unit circle, being traced out counterclockwise as θ grows. But this means that any complex number $a+bi$ can be rewritten in the form $re^{i\theta}$, where r is the radius of the origin-centered circle that it's on (or in other words, r is the distance from the origin), and θ is the angle formed by measuring counterclockwise from the positive x-axis. Notice that the choice of θ is not unique for a point in the plane: if you add or subtract 2π from θ, you go all the way around the circle and come back to the same place.

We'll call $a + bi$ *rectangular coordinates* to describe a point, and we'll call $re^{i\theta}$ *polar coordinates.* It's not hard to write down the equations that go back and forth between these two forms: $r = \sqrt{a^2 + b^2}$ because it's the distance from the origin to (a, b). And we can see from forming a basic triangle that $\tan\theta = b/a$.

In the other direction, it's even simpler, because (a, b) is on the circle of radius r at angle θ, so $a = r\cos\theta$, $b = r\sin\theta$.

Example 4.

- *Put* $3+4i$ *in polar coordinates.* We have $r = \sqrt{9+16} = \sqrt{25} = 5$, and $\theta = \arctan(4/3)$, which turns out to be about 0.93 radians. Thus $3 + 4i \approx 5\, e^{0.93i}$.
- *Put* $7e^{i\frac{\pi}{2}}$ *in rectangular coordinates.* We get $a = 7\cos(\pi/2) = 0$ and $b = 7\sin(\pi/2) = 7$, so $7e^{i\frac{\pi}{2}} = 7i$.

5. Finding nth roots

We've seen that the new notation i is built to take the square root of -1, but we'll finish these notes by observing that complex numbers let us take all nth roots of all real or complex numbers!

Suppose we want to find the nth root of some number w. That means solving the equation $z^n = w$. Suppose $z = se^{i\alpha}$ and $w = re^{i\theta}$. Then by De Moivre's theorem, we have

$$\left[se^{i\alpha}\right]^n = re^{i\theta} \iff s^n e^{in\alpha} = re^{i\theta}.$$

Now if two complex numbers in polar form are equal, they must have the same distance from the origin and their angles must differ by a multiple of 2π. (Can you see why?)

This gives $s^n = r$ and $in\alpha = i(\theta + 2\pi k)$. Remembering that $r, s > 0$, the first equation has an easy solution, $s = \sqrt[n]{r}$. Solving the second equation, we get $\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}$, for $k = 0, 1, 2, \ldots, n-1$. Summary:

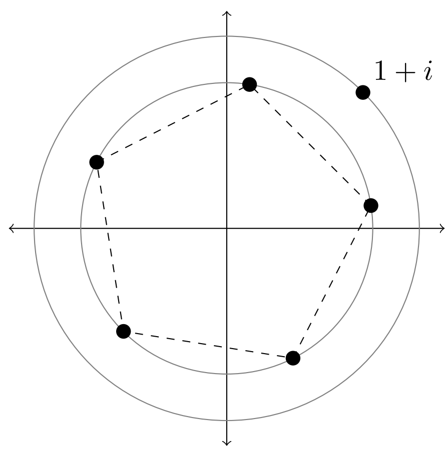
$$\boxed{\sqrt[n]{re^{i\theta}} = \sqrt[n]{r} \cdot e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)}, \qquad k = 0, 1, \ldots, n-1}$$

This tells us that just like in the real numbers, where there are two different possible square roots of 9, say, in the complex numbers there are n different possible nth roots of any nonzero number. And we find them in the following simple way: take the nth root of the radial coordinate, and divide the angle by n, then use the n evenly spaced angles around the circle that include that answer. A nice way to visualize this is that the square roots form the vertices of a regular n-sided polygon centered at the origin!

Example 5. Let's check the method by finding $\sqrt{-1}$. We know $-1 = e^{i\pi}$. Here, $r = 1$ and $\theta = \pi$. So we take the square root of r, getting 1. (Important note: r values are always positive real, so we take the positive real square root of 1, which is just 1.) And we chop the angle π in half, getting $\pi/2$. Thus one square root of -1 is $1 \cdot e^{i\frac{\pi}{2}}$, which is i, just as we expected. And the other one is at an angle evenly spaced around the circle from this one, which means the other square root is $-i$.

Example 6. *Find the fifth roots of $w = 1 + i$.* First we represent w in polar form, as $w = \sqrt{2} \cdot e^{i\frac{\pi}{4}}$. (We got this because $r = \sqrt{1^2 + 1^2}$ and $\theta = \arctan 1$.) Then we take $(\sqrt{2})^{1/5} = \sqrt[10]{2}$ to get our r value, $\frac{\pi/4}{5} = \frac{\pi}{20}$ to get our first θ value, and $\frac{2\pi}{5} = \frac{8\pi}{20}$ to see how the angles are spaced out. So finally, the fifth roots are the numbers

$$z = \sqrt[10]{2} \cdot e^{i\frac{\pi}{20}}, \quad \sqrt[10]{2} \cdot e^{i\frac{9\pi}{20}}, \quad \sqrt[10]{2} \cdot e^{i\frac{17\pi}{20}}, \quad \sqrt[10]{2} \cdot e^{i\frac{25\pi}{20}}, \quad \sqrt[10]{2} \cdot e^{i\frac{33\pi}{20}}.$$



Fifth roots of $1 + i$