Final Exam Math 34 (Calc II), Spring 2013

SOLUTIONS

Name, Section

No notes or devices with calculator functionality. Turn your phone off!

Please ask for scratch paper if needed.

You must evaluate standard trig values, but you do not need to simplify your answers otherwise.

Except for multiple choice, you must show work to receive full credit.

You must verify the hypotheses for any convergence test that you use.

If you claim that an inequality holds, you must justify that claim except if you judge it to be obvious. For example, for $n = 1, 2, 3, \ldots$, the inequality $n < 3n + 2$ is obvious, while $2(n + 10)! \leq (2n)!$ is not.

Classical series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for all } x$$

$$\ln(x) = -(1-x) - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \cdots - \frac{(1-x)^k}{k} + \cdots = -\sum_{k=1}^{\infty} \frac{(1-x)^k}{k} \quad \text{for } 0 < x < 2$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \quad \text{for all } x$$

$$\arctan x = 1 - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \text{for } |x| \leq 1$$

$$(1+x)^p = 1 + px + \tfrac{p(p-1)}{2\cdot 1} x^2 + \cdots + \tfrac{p(p-1)\cdots(p-k+1)}{k!} x^k + \cdots = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \quad \text{for } |x| < 1 \text{ and any real } p$$

I pledge that I have neither given nor received assistance on this test.

____

(signature)

(1) (16 pts) Evaluate each integral, or show it diverges.

(a) $\displaystyle\int_0^{\pi} x\sin(4x)\,dx$

INTEGRATION BY PARTS: $\displaystyle\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

$u = x,\quad dv = \sin(4x)\,dx$

$du = dx,\quad v = -\frac{1}{4}\cos(4x)$

$$-\frac{1}{4}x\cos(4x)\Big|_0^{\pi} + \frac{1}{4}\int_0^{\pi}\cos(4x)\,dx$$

$$= \Big(-\frac{1}{4}\pi\,\underset{\cos(4\pi)}{(1)} + \frac{1}{4}\cdot 0\Big) + \frac{1}{4}\Big[\underbrace{\frac{1}{4}\sin(4x)\Big|_0^{\pi}}_{0-0}\Big] = \frac{-\pi}{4}.$$

(b) $\displaystyle\int \frac{4x+2}{x^2+x-2}\,dx$

PARTIAL FRACTIONS

$x^2+x-2 = (x+2)(x-1)$

$$\frac{A}{x+2} + \frac{B}{x-1} = \frac{4x+2}{x^2+x-2}$$

solve:

$A(x-1) + B(x+2) = 4x+2$

$\Downarrow$

$(A+B)x + (-A+2B) = 4x+2$

obtain:

$A + B = 4$

$-A + 2B = 2$

$3B = 6$

$\Downarrow$

$B = 2$, so $A = 2$

integral becomes

$$\int\Big(\frac{2}{x+2} + \frac{2}{x-1}\Big)dx$$

$$= 2\ln|x+2| + 2\ln|x-1| + C$$

Alternate approach: u-SUBSTITUTION

$u = x^2+x-2$

$du = (2x+1)\,dx$

$$\int\frac{4x+2}{x^2+x-2}\,dx = \int\frac{2\,du}{u}$$

$$= 2\ln|u| + C = 2\ln|x^2+x-2| + C$$

Same answer!

(c) $\int_{100}^{\infty} \cos(x)\, dx$

$$\int_{100}^{\infty} \cos x\, dx = \lim_{b \to \infty} \left(\int_{100}^{b} \cos x\, dx \right)$$

$$= \lim_{b \to \infty} \left(\sin x \Big|_{100}^{b} \right) = \lim_{b \to \infty} \left(\sin b - \sin 100 \right)$$

but $\lim_{b \to \infty} \sin b$ DNE (oscillation)

So this improper integral is <u>divergent</u>.

(the signed area oscillates!)

(d) $\int \cos^3(x)\, dx$

$\sin^2 x + \cos^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$.

Thus

$$\int \cos^3 x\, dx = \int (1 - \sin^2 x)\, \underline{\cos x\, dx} = \int (1 - u^2)\, du$$

$u = \sin x$

$du = \cos x\, dx$

$$= u - \frac{1}{3} u^3 + C$$

$$= \sin x - \frac{1}{3} \sin^3 x + C.$$

(2) (16 pts) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n(\ln n)^2}$ (note typo: sum should start at $n=2$ or else it's undefined)

This is an alternating series. First test for absolute convergence using INTEGRAL TEST.

(hypotheses: associated function $f(x) = \frac{1}{x(\ln x)^2}$ is continuous ✓ positive ✓ and clearly decreasing since x, $\ln x$ are increasing ✓)

So our sum converges if and only if the improper integral $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$ does.

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\,du = \int_{\ln 2}^{\infty} u^{-2}\,du =$$

$$\begin{pmatrix} u = \ln x \\ du = \frac{1}{x}dx \end{pmatrix}$$

$$= \lim_{b\to\infty}\left[-\frac{1}{u}\Big|_{\ln 2}^{b} \right]$$

$$= \lim_{b\to\infty}\left(\cancel{-\frac{1}{b}}^{\,0} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$$

So the original series is ABSOLUTELY CONVERGENT.

(b) $\sum_{k=1}^{\infty} \frac{\sin k}{k^{10}}$

To test for absolute convergence, consider

$\sum_{k=1}^{\infty} \frac{|\sin k|}{k^{10}}$. We can apply DIRECT/ORDINARY COMPARISON TEST

(hypothesis: terms are positive ✓)

with $\frac{|\sin k|}{k^{10}} < \frac{1}{k^{10}}$.

We know $\sum_{k=1}^{\infty} \frac{1}{k^{10}}$ converges (it's a p-series with $p=10$, and these converge when $p>1$)

so $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^{10}}$ does too.

Original series is ABSOLUTELY CONVERGENT.

(3) (12 pts) (a) Let $f(x) = \sum_{n=0}^{\infty} x^n$. Find series representations for $f'(x)$ and $\int f(x)\,dx$.

$$f'(x) = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} + C$$

(b) What real number does the series $\sum_{n=1}^{\infty} n\left(\frac{1}{5}\right)^{n-1}$ sum to?

From above, this is $f'(\frac{1}{5})$ where $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$,

From knowledge of derivatives, $(|x|<1\ \checkmark)$

we have $f'(x) = \frac{1}{(1-x)^2}$, so $f'(\frac{1}{5}) = \frac{1}{(4/5)^2} = \frac{25}{16}$.

(c) What real number does the series $\sum_{n=1}^{\infty} \frac{n \cdot 5^n}{6^{n+1}}$ sum to?

$$\sum_{n=1}^{\infty} \frac{n\,5^n}{6^{n+1}} = \frac{1}{6}\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^n = \left(\frac{1}{6}\cdot\frac{5}{6}\right)\underbrace{\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}}$$

as above, this is

$f'(\frac{5}{6}) = \frac{1}{(1/6)^2} = 36$

$(\frac{5}{6} < 1\ \checkmark)$

Final answer:

$\frac{1}{6}\cdot\frac{5}{6}\cdot 36 = 5$

(4) (8 pts) Find the interval of convergence and the radius of convergence for $\sum_{n=1}^{\infty} \frac{(-1)^n(x-3)^n}{5^n \cdot n^2}$.

We'll use RATIO TEST FOR ABSOLUTE CONVERGENCE

with $a_n = \frac{|x-3|^n}{5^n \cdot n^2}$ (hypothesis : terms positive ✓)

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x-3|^{n+1} / 5^{n+1}(n+1)^2}{|x-3|^n / 5^n n^2}$$

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1$$

$$= \lim_{n\to\infty} \underbrace{\cancel{\frac{n^2}{(n+1)^2}}}_{1} \cdot \frac{|x-3|}{5} = \frac{|x-3|}{5}$$

So $r<1 \iff \frac{|x-3|}{5} < 1 \iff |x-3| < 5$

$$\underbrace{-2 \quad\quad 3 \quad\quad 8}$$

Radius of convergence $= 5$

Test endpoints:

$(x=-2)$ $\sum \frac{(-1)^n(-5)^n}{5^n n^2} = \sum \frac{1}{n^2}$ ✓ (p-series, $p=2>1$)

$(x=8)$ $\sum \frac{(-1)^n 5^n}{5^n n^2} = \sum \frac{(-1)^n}{n^2}$ ✓ (alt. p-series)

Interval of convergence : $[-2, 8]$

(5) (10 pts) (a) Find the second-degree Taylor polynomial for $f(x) = (1+x)^{3/4}$ centered at $x = 0$.

$$P_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(a)}{n!}(x-a)^n \; ; \quad \text{here } a = 0.$$

$$f(x) = (1+x)^{3/4} \; ; \quad f(0) = 1$$

$$f'(x) = \frac{3}{4}(1+x)^{-1/4} \; ; \quad f'(0) = 3/4$$

$$f''(x) = \frac{-3}{16}(1+x)^{-5/4} \; ; \quad f''(0) = \frac{-3}{16}$$

$$P_2(x) = 1 + \frac{3/4}{1!}x^1 + \frac{-3/16}{2!}x^2$$

$$= 1 + \frac{3}{4}x - \frac{3}{32}x^2.$$

(b) Find the Taylor series for $g(x) = e^{-4x}$ centered at $a = -2$. Write your answer in Σ notation.

$$g(x) = e^{-4x} \; ; \quad g(-2) = e^8$$

$$g'(x) = -4e^{-4x} \; ; \quad g'(-2) = -4e^8$$

$$g''(x) = 16e^{-4x} \; ; \quad g''(-2) = 16e^8$$

$$\vdots$$

$$g^{(n)}(x) = (-4)^n e^{-4x} \; ; \quad g^{(n)}(-2) = (-4)^n e^8$$

Taylor series: $\displaystyle\sum_{n=0}^{\infty} \frac{g^{(n)}(-2)}{n!}(x+2)^n = \sum_{n=0}^{\infty} \frac{(-4)^n e^8}{n!}(x+2)^n$

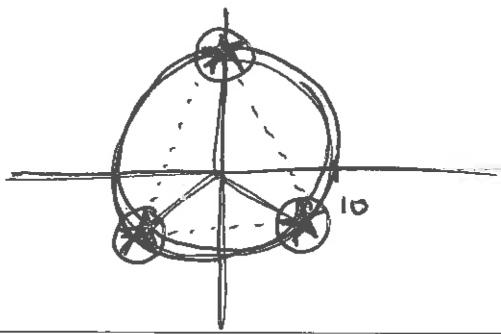
(6) (10 pts) (a) Find the cube roots of $-1000i$ in polar coordinates and plot them.

Put $-1000i$ in polar: $-1000i = re^{i\theta}$

if $r = 1000$, $\theta = \frac{3\pi}{2}$

pos. $\left.\begin{array}{l} \sqrt[3]{r} = \sqrt[3]{1000} = 10 \\ \frac{\theta}{3} = \frac{\pi}{2} \end{array}\right\}$ so cube roots are

$10e^{i\frac{\pi}{2}}$, $10e^{i(\frac{\pi}{2}+\frac{2\pi}{3})}$, $10e^{i(\frac{\pi}{2}+\frac{4\pi}{3})}$

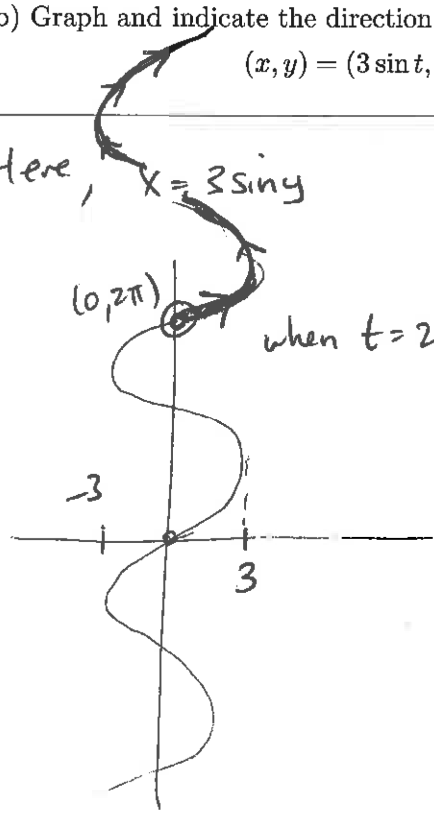


(b) Graph and indicate the direction of increasing t:

$$(x, y) = (3\sin t,\ t) \qquad \text{for} \quad 2\pi < t < \infty.$$

Here, $x = 3\sin y$

when $t = 2\pi$, $(x,y) = (0, 2\pi)$



(7) (28 pts) **Multiple choice.** Circle ALL correct answers. Show work for partial credit.

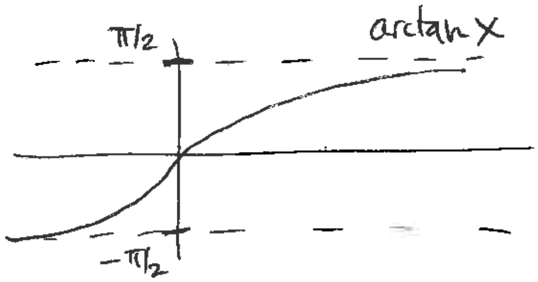
(a) The sequence $a_n = \arctan(n^2)$ for $n \geq 1$...

(i) is a bounded monotone sequence;

(ii) converges to $\pi/2$;

~~(iii)~~ diverges by oscillation;

~~(iv)~~ is decreasing.



(b) The series $\displaystyle\sum_{k=4}^{\infty} \frac{3^k}{(k+1)^k}$ ...

(i) converges by the root test;

~~(ii)~~ diverges by the root test;

~~(iii)~~ makes the root test inconclusive;

~~(iv)~~ is a geometric series;

(v) converges by comparison to $\sum \left(\frac{3}{4}\right)^k$.

(c) The classical series expansion for $\ln(x)$ ...

(see table on front page of exam)

~~(i)~~ has only even powers of x;

(ii) converges at $x = 0.2$;

~~(iii)~~ can be used to approximate $\ln(15)$;

(iv) sometimes gives an alternating series;

~~(v)~~ is a Maclaurin series as well as a Taylor series.

$$\ln x = -\sum_{k=1}^{\infty} \frac{(1-x)^k}{k} \quad \text{for } 0<x<2$$

(alternating when $1-x<0$
i.e. $x>1$)

(d) Which of these $\binom{p}{k}$ expressions are correct?

(i) $\binom{5}{2} = \binom{5}{3}$;

(ii) $\binom{-1}{2} < 0$;

(iii) $\binom{-1}{3} < 0$;

~~(iv)~~ $\binom{1/2}{2} = -\frac{1}{9}$.

Recall $\binom{p}{k} = \dfrac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$ $\Big\}$ k terms on top + bottom

$\binom{5}{2} = \dfrac{5\cdot 4}{2}$ $\quad \binom{5}{3} = \dfrac{5\cdot 4\cdot 3}{3\cdot 2}$

$\binom{-1}{2} = \dfrac{(-1)(-2)}{2}$; $\quad \binom{-1}{3} = \dfrac{(-1)(-2)(-3)}{3\cdot 2}$; $\quad \binom{1/2}{2} = \dfrac{(1/2)(-1/2)}{2} = \dfrac{-1}{8}$

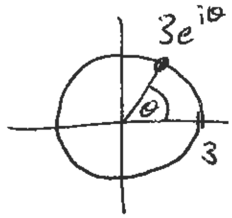
(e) The complex numbers $3e^{i\theta}$ for $-\infty < \theta < \infty$ ...

~~(i)~~ have rectangular coordinates satisfying $x^2 + y^2 = 3$;

(ii) have real part $3\cos\theta$;

(iii) graph as a circle in the complex plane;

~~(iv)~~ graph as a spiral in the complex plane.



(f) The formula $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

(i) can be derived from the Taylor series for arctan;

~~(ii)~~ is a convergent p-series;

~~(iii)~~ is a divergent p-series;

(iv) only requires 2 terms to approximate $\frac{\pi}{4}$ within 0.25.

$$\arctan x = 1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

so $\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$\frac{\pi}{4} \approx 1 - \frac{1}{3}$ is accurate within $\frac{1}{5} = .2$

because $R_n \leq a_{n+1}$ for alternating series

1	2	3	4	5	6	7	TOTAL
[16]	[16]	[12]	[8]	[10]	[10]	[28]	**[100]**