KEY

Math 34

Tufts University
Department of Mathematics
Final Exam

December 13, 2013
8:30-10:30am

No books, notes or calculators are allowed. Cross out what you do not want us to grade. You must show all your work in order to receive full credit unless otherwise indicated. You may **not** refer to growth rates when taking limits. You are expected to evaluate all trigonometric values. Please write neatly. If you use the scratch work page at the end of the exam for work you want graded, please make sure you indicate that. You are required to sign your exam. With your signature, you pledge that you have neither given nor received assistance on this exam.

Problem	Point Value	Points
1	10	
2	10	
3	10	
4	10	
5	8	
6	18	
7	8	
8	8	
9	8	
10	10	
	100	

1. (10 points) Evaluate the following integrals. **Do not simplify your answers.**

(a) $\displaystyle\int \frac{3x^2+5x+3}{x^3+x}\,dx \quad = \quad \int \frac{3x^2+5x+3}{x(x^2+1)}\,dx$

PF's $\quad \dfrac{3x^2+5x+3}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

(*) $3x^2+5x+3 = A(x^2+1) + (Bx+C)x$

$x=0 \qquad 3 = A$

$x^2: \qquad 3 = A+B \Rightarrow B=0$

$x: \qquad 5 = C$

$$\int \frac{3dx}{x} + \int \frac{5dx}{x^2+1}$$

$$= \boxed{3\ln|x| + 5\tan^{-1}x + C}$$

(b) $\displaystyle\int \frac{dx}{(x^2+1)^2}$ (Note: Do *not* try to use partial fractions here.)

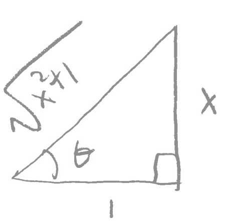
$x = \tan\theta, \; dx = \sec^2\theta\, d\theta$

$(x^2+1)^2 = (\tan^2\theta+1)^2 = (\sec^2\theta)^2 = \sec^4\theta$

$$= \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta\, d\theta = \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = \frac{1}{2}\left[\theta + \frac{1}{2}(2\sin\theta\cos\theta)\right] + C$$

$$= \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$



$\tan\theta = x$

$$= \boxed{\frac{1}{2}\left(\tan^{-1}x + \frac{x}{\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}}\right) + C}$$

2. (10 points)

Determine whether the following series converge or diverge. Justify your answers. *If the series converges find its sum.*

(a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, 2^k}{3^{k-1}}$ $= \frac{2}{3^0} - \frac{2^2}{3^1} + \frac{2^3}{3^2} + \cdots$

This is a geometric series with $r = -2/3$.

Since $|r| = |-2/3| = 2/3 < 1$, the series c's to $S = \frac{a}{1-r}$

$$= \frac{2}{1-(-2/3)} = \frac{2}{5/3} = \boxed{\frac{6}{5}}$$

(b) $\sum_{k=1}^{\infty} \frac{1}{1+\left(\frac{1}{2}\right)^k}$

Divergence test. $\lim_{k\to\infty} \frac{1}{1+(\frac{1}{2})^k} = 1 \neq 0$

$(\frac{1}{2})^k \downarrow 0$ since $|\frac{1}{2}|<1$

$\therefore \Sigma$ D's by Div. Test.

3. (10 points) Determine whether each of following series is absolutely convergent (AbC), conditionally convergent (CC) or divergent (D). Justify your answers.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{3 \cdot n!}$

$\boxed{1}$ AbC? $\sum_{n=0}^{\infty} \frac{1}{3 \cdot n!}$

Ratio Test: $r = \lim_{n\to\infty} \frac{\frac{1}{3(n+1)!}}{\frac{1}{3\,n!}} = \lim_{n\to\infty} \frac{3\,n!}{3(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)\cdot n!}$

$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

Σ is AbC

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{8n+1}$

$\boxed{1}$ AbC? $\sum_{n=1}^{\infty} \frac{1}{8n+1}$ positive Σ - LCT with $\Sigma \frac{1}{n}$

$L = \lim_{n\to\infty} \frac{\left(\frac{1}{8n+1}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{8n+1} = \lim_{n\to\infty} \frac{1}{8+\frac{1}{n}} \xrightarrow{} 0 = \frac{1}{8}$

Since $0 < \frac{1}{8} < \infty$ and since $\Sigma \frac{1}{n}$ is the divergent harmonic series, our series is NOT AbC.

$\boxed{2}$ CC? $\sum_{n=1}^{\infty} \frac{(-1)^n}{8n+1}$ A.S. Test. ① $\lim_{n\to\infty} \frac{1}{8n+1} = 0$ ✓

② $\frac{1}{8n+1} > 0$ for $n \geq 1$ ✓

③ $\frac{1}{8(n+1)+1} \leq \frac{1}{8n+1}$, $n \geq 1$ ✓

$\therefore \Sigma$ passes A.S. Test + is CC

4. (10 points) Find the third-order Taylor polynomial $p_3(x)$ for each of the following functions.

(a) $f(x) = x \ln x$, centered at $a = 1$.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x \ln x$	0
1	$\ln x + x\left(\frac{1}{x}\right) = \ln x + 1$	1
2	$\frac{1}{x}$	1
3	$-\frac{1}{x^2}$	-1

$$p_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$\boxed{p_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3}$$

(b) $g(x) = 4\sin(2x)$ centered at $a = \pi/8$.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/8)$
0	$4\sin(2x)$	$4\sin \pi/4 = 4(\sqrt{2}/2) = 2\sqrt{2}$
1	$8\cos(2x)$	$8\cos \pi/4 = 8(\sqrt{2}/2) = 4\sqrt{2}$
2	$-16\sin(2x)$	$-16\sin \pi/4 = -16(\sqrt{2}/2) = -8\sqrt{2}$
3	$-32\cos(2x)$	$-32\cos \pi/4 = -32(\sqrt{2}/2) = -16\sqrt{2}$

$$p_3(x) = f(\pi/8) + f'(\pi/8)(x-\pi/8) + \frac{f''(\pi/8)}{2!}(x-\pi/8)^2 + \frac{f^{(3)}(\pi/8)}{3!}(x-\pi/8)^3$$

$$p_3(x) = 2\sqrt{2} + 4\sqrt{2}(x-\pi/8) + \frac{-8\sqrt{2}}{2}(x-\pi/8)^2 + \frac{-16\sqrt{2}}{6}(x-\pi/8)^3$$

$$\boxed{p_3(x) = 2\sqrt{2} + 4\sqrt{2}(x-\pi/8) - 4\sqrt{2}(x-\pi/8)^2 - \frac{8\sqrt{2}}{3}(x-\pi/8)^3}$$

5. (8 points) Consider the following parametric equations:

$$x = 4\cos^2 t \text{ and } y = 4\sin^2 t \quad \pi/2 \le t \le \pi$$

(a) Eliminate the parameter t to obtain an equation in x and y.

$\frac{x}{4} = \cos^2 t \qquad \frac{y}{4} = \sin^2 t$

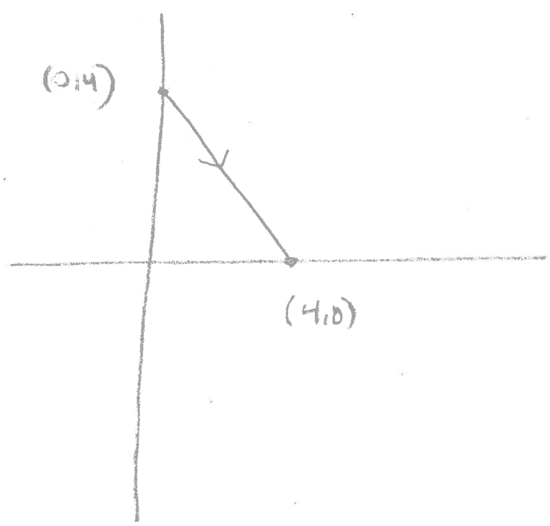
$\cos^2 t + \sin^2 t = 1$

$\frac{x}{4} + \frac{y}{4} = 1$

$\boxed{x + y = 4}$

(b) Sketch the curve and indicate the positive orientation (direction of increasing t).

t	$x = 4\cos^2 t$	$y = 4\sin^2 t$
$\frac{\pi}{2}$	0	4
$\frac{3\pi}{4}$	2	2
π	4	0



6. You may use any of the following.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

(a) (4 points) Find a series representation for $f(x) = \ln(1+x/2)$ and its interval of convergence.

$$\ln\left(1+\frac{x}{2}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\left(\frac{x}{2}\right)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{2^k k}$$

for $-1 < \frac{x}{2} \le 1$

$-2 < x \le 2$

(b) (4 points) Use series to evaluate $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

$$\lim_{x\to 0} \frac{\left(x - x^3/3! + x^5/5! - x^7/7! + \cdots\right) - x}{x^3}$$

$$= \lim_{x\to 0} \frac{-x^3/3! + x^5/5! - x^7/7! + \cdots}{x^3}$$

$$= \lim_{x\to 0} \left(-1/3! + \underbrace{x^2/5! - x^4/7! + \cdots}_{}\right) = -\frac{1}{3!} = \boxed{-\frac{1}{6}}$$

$$\left(\text{check: } \lim_{x\to 0} \frac{\sin x - x \to 0}{x^3 \to 0} \overset{L}{=} \lim_{x\to 0} \frac{\cos x - 1 \to 0}{3x^2 \to 0} \overset{L}{=} \lim_{x\to 0} \frac{-\sin x \to 0}{6x \to 0} \overset{L}{=} \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}\right)$$

9. (8 points) Determine whether the following sequences converge or diverge. If the sequence converges find its limit. Answers only will be graded. No partial credit.

(a) $a_n = 1^n$

$\lim_{n\to\infty} 1^n = 1$ $\boxed{\text{C's to } 1}$

(b) $a_n = \dfrac{1}{tan^{-1}n}$

$\lim_{n\to\infty} \dfrac{1}{\tan^{-1} n} = \dfrac{1}{\pi/2} = \dfrac{2}{\pi}$ $\boxed{\text{C's to } \frac{2}{\pi}}$

(c) $a_n = 2a_{n-1}, n \geq 1;\ \ a_0 = 3$

$a_n = 3 \cdot 2^n$

$a_1 = 2a_0$

$a_2 = 2^2 a_0$

$a_3 = 2^3 a_0$

$\vdots$

$a_n = 2^n \cdot a_0$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3 \cdot 2^n \to \infty$

$\boxed{\text{Seq. D's}}$

(d) $a_n = \dfrac{(\ln n)^{10}}{n}$

$\lim_{n\to\infty} \dfrac{(\ln n)^{10} \to \infty}{n \to \infty} = \lim_{x\to\infty} \dfrac{(\ln x)^{10} \to \infty}{x \to \infty} = \lim_{x\to\infty} \dfrac{10 (\ln x)^9}{x} = \cdots = \lim_{x\to\infty} \dfrac{10!}{x} = 0$

$\boxed{\text{Seq. C's to } 0}$

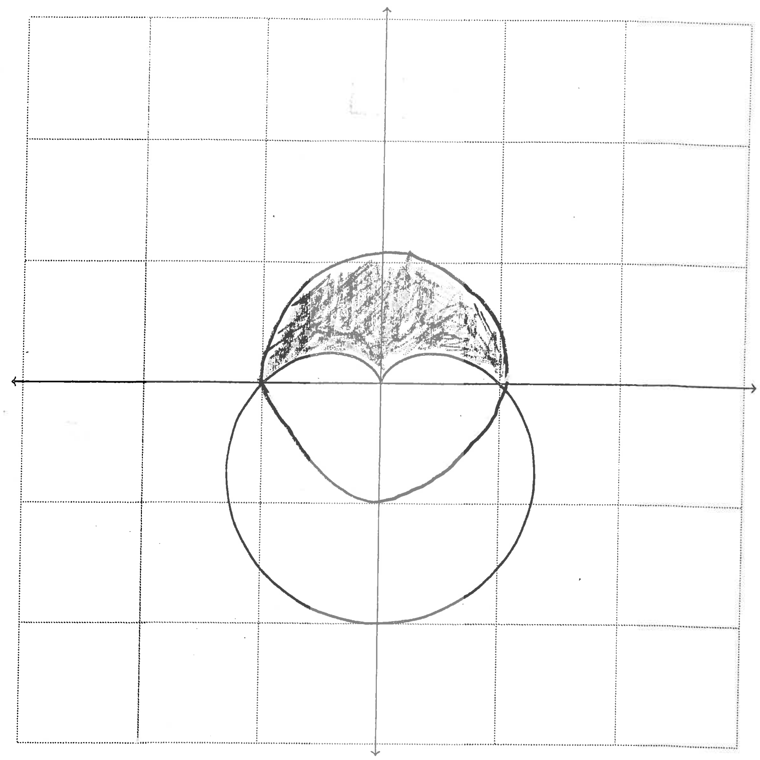
8. (8 points) Polar coordinates.

(a) Below is a sketch of the cardioid $r = 1 - \sin\theta$. On the same polar graph sketch the circle $r = 1$ and shade the region inside the circle and outside the cardioid.

$1 = 1 - \sin\theta$

$\sin\theta = 0$ ✓

$\theta = 0, \pi$



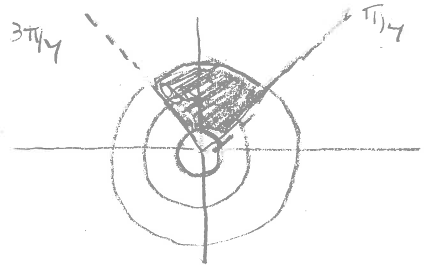
(b) Set up but *do not evaluate* a definite integral that expresses the area of the region you shaded in part(*a*).

$$A = \frac{1}{2}\int_0^{\pi} (R^2 - r^2)\,d\theta = \boxed{\frac{1}{2}\int_0^{\pi} \left(1^2 - (1-\sin\theta)^2\right)d\theta}$$

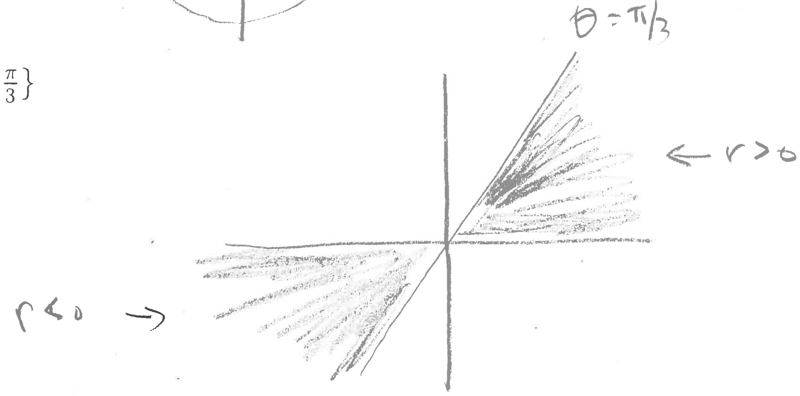
7. (8 points) Polar coordinates.

(a) Sketch each of the following sets of points in the polar plane. (Use separate sketches for each set.)

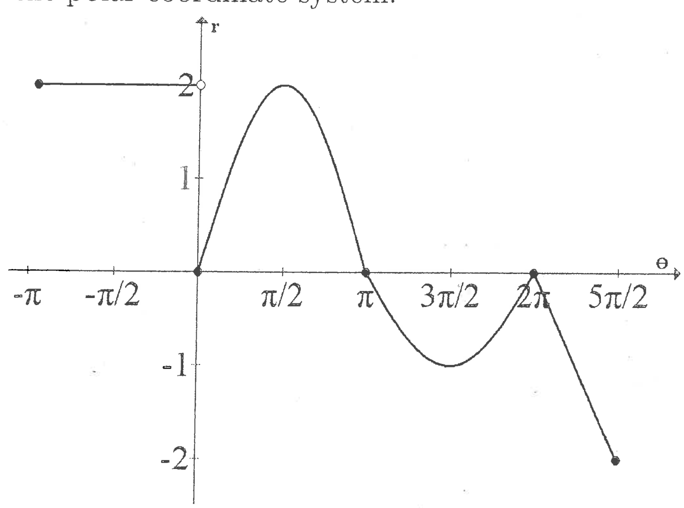
i. $\{(r,\theta) : 1 \leq r \leq 3 \text{ and } \frac{\pi}{4} < \theta < \frac{3\pi}{4}\}$

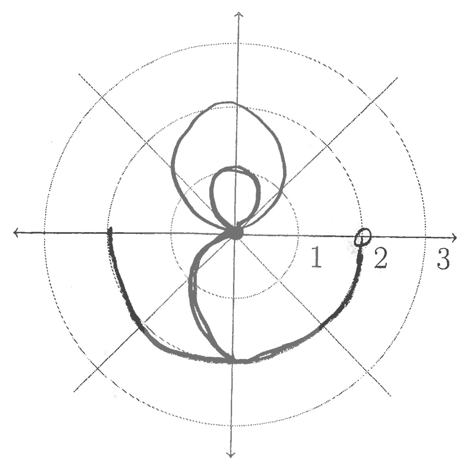


ii. $\{(r,\theta) : 0 \leq \theta \leq \frac{\pi}{3}\}$



(b) The following curve is an (r,θ) Cartesian graph of a polar curve. Sketch it on the right using the polar coordinate system.





You may use any of the following.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \leq 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

(c) (2 points) Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!}$

$$= \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

(d) (8 points)Approximate the following definite integral to within an accuracy of $\frac{1}{1000}$.

$$\int_0^{\frac{1}{10}} \cos\sqrt{x}\, dx$$

$$\int_0^{1/10} \left(1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots\right) dx$$

$$= \int_0^{1/10} \left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right)$$

$$= \left[x - \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 4!} - \frac{x^4}{4 \cdot 6!} + \cdots\right]_0^{\frac{1}{10}}$$

$$= \frac{1}{10} - \frac{\left(\frac{1}{10}\right)^2}{2 \cdot 2!} + \frac{\left(\frac{1}{10}\right)^3}{3 \cdot 4!} - \frac{\left(\frac{1}{10}\right)^4}{4 \cdot 6!} + \cdots$$

$$= \frac{1}{10} - \frac{1}{100(4)} + \frac{1}{(1000)(3) \cdot 4!} + \frac{1}{10^4 \cdot 4 \cdot 6!} - \cdots$$

A.S. Estimation:

$$\frac{1}{1000 \cdot 3 \cdot 4!} < \frac{1}{1000} \quad \text{So}$$

$$\int_0^{1/10} \cos\sqrt{x}\, dx$$

$$\approx \frac{1}{10} - \frac{1}{400}$$

$$\approx \frac{40-1}{400} = \boxed{\frac{39}{400}}$$

10. (10 pts) **True/false questions.** Decide whether each of the following statements is true or false. Indicate your answer by shading the corresponding box. No partial credit.

(a) A convergent positive series $\sum a_n$ is absolutely convergent. T F

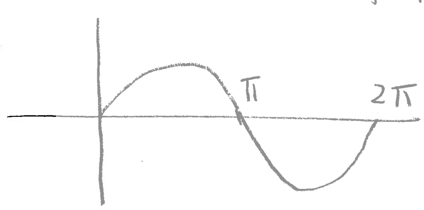
(b) The graph of $r \sin\theta = 3$ in the polar plane is a horizontal line. T F

$y = 3$

(c) The graph of the first Taylor polynomial of $f(x)$ centered at $x = 3$ is simply the tangent line to $f(x)$ at the point $(3, f(3))$. T F

(d) $\sum \sin n$ is an alternating series. T F

+ + +, −, −, − etc



(e) $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx = 0$ T F

Improper!!

$$\int_0^1 \frac{dx}{x^2} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^2}$$

$$= \lim_{t\to 0^+} -\frac{1}{x}\Big|_t^1 \quad \lim_{t\to 0^+}\left[-1 + \frac{1}{t}\right]$$

$\downarrow$

∞ D's

Scratch Work