Math 12 Final Exam Solutions Fall 2010

1 (18) (a) $\int \cos^4(5x)\,dx = \int [\cos^2 5x]^2 dx = \int \left(\frac{1}{2}(1+\cos 10x)\right)^2 dx$

$= \frac{1}{4}\int (1 + 2\cos 10x + \cos^2 10x)\,dx = \frac{1}{4}\int \left(1 + 2\cos 10x + \frac{1}{2}(1+\cos 20x)\right) dx$

$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 10x + \frac{1}{2}\cos 20x\right) dx = \boxed{\frac{1}{4}\left[\frac{3}{2}x + \frac{2}{10}\sin 10x + \frac{1}{40}\sin 20x\right] + C}$

(b) $\int \frac{dx}{x^2(x-1)} = -\int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{dx}{x-1}$

PF: $\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

$\Rightarrow \quad 1 = Ax(x-1) + B(x-1) + Cx^2$

$x=0 \quad 1 = B(-1) \quad B=-1$

$x=1 \quad 1 = C$

$0\cdot x^2 = Ax^2 + Cx^2$

$0 = A+1 \Rightarrow A=-1$

$= \boxed{-\ln|x| + \frac{1}{x} + \ln|x-1| + C}$

2 (12) (a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Integral Test.

Let $f(x) = \frac{1}{x(\ln x)^2}$, $x \geq 2$

Then f is continuous and positive $n \geq 2$.

$f'(x) = -1\left[(x)(\ln x)^2\right]^{-2}\left[1\cdot(\ln x)^2 + x\cdot 2(\ln x)\frac{1}{x}\right] < 0$

$- \quad + \quad + \quad +$

for $x \geq 2$ so f is decreasing.

$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty}\int_2^t \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty}\left[-\frac{1}{\ln x}\right]_2^t$

$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C$

$u = \ln x,\ du = \frac{1}{x}dx \quad = -\frac{1}{\ln x} + C$

$= \lim_{t\to\infty}\left(\underbrace{-\frac{1}{\ln t}}_{\to 0} + \frac{1}{\ln 2}\right) = 1/\ln 2$

$\therefore$ The integral C's $\Rightarrow \sum$ C's by the integral test.

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{15n^3+3n+1} \sim \sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$

LCT $L = \lim_{n\to\infty} \frac{\sqrt{n^2+3}}{15n^3+3n+1}\cdot\frac{n^3}{\sqrt{n^2}} = \lim_{n\to\infty} \underbrace{\sqrt{\frac{n^2+3}{n^2}}}_{\to 1}\cdot\underbrace{\frac{n^3}{15n^3+3n+1}}_{\to 1/15}$

$L = 1/15$

(c) $\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{2\tan\theta\, 2\sec\theta\tan\theta\,d\theta}{2\sec\theta}$

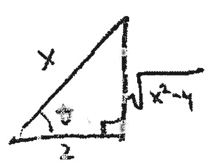
$\boxed{x = 2\sec\theta}$ θ:

$dx = 2\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4\tan^2\theta} = 2\tan\theta$

$= 2\int \tan^2\theta\,d\theta = 2\int(\sec^2\theta - 1)\,d\theta$

$= 2\left[\tan\theta - \theta\right] + C$



$= \boxed{2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C}$

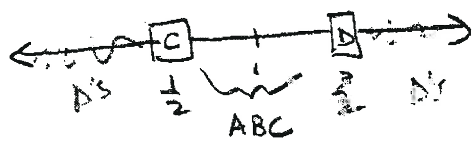
$\sec\theta = \frac{x}{2}$

Since $0 < \frac{1}{15} < +\infty$ and since $\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent Harmonic $\sum$ our $\sum$ D's by LCT.

$\boxed{3}$ (10) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{n}$ RATFACE (Center = 1)

(i) $$\rho = \lim_{n\to\infty} \left| \frac{2^{n+1}(x-1)^{n+1}}{(n+1)} \cdot \frac{n}{2^n (x-1)^n} \right| = 2|x-1| \lim_{n\to\infty} \overbrace{\frac{n}{n+1}}^{\to 1} = 2|x-1|$$

(ii) $\rho < 1$: $$2|x-1| < 1 \iff |x-1| < \tfrac{1}{2} \iff -\tfrac{1}{2} < x-1 < \tfrac{1}{2} \iff \tfrac{1}{2} < x < \tfrac{3}{2}$$



(iii) endpoints: $x = \frac{1}{2}$ $$\sum_{n=1}^{\infty} \frac{2^n(\frac{1}{2}-1)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n(-\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ Alt. H-Σ. C's

$x = \frac{3}{2}$ $$\sum_{n=1}^{\infty} \frac{2^n(\frac{3}{2}-1)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n(\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$ H-Σ D's

$$\boxed{ROC = \tfrac{1}{2} \qquad IOC = [\tfrac{1}{2}, \tfrac{3}{2})}$$

$\boxed{4}$ (10) (a) $$f(x) = \frac{\tan^{-1}(4x^2)}{x^2} = \sum_{n=0}^{\infty} \frac{1}{x^2}\left[\frac{(-1)^n (4x^2)^{2n+1}}{(2n+1)}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{4n+2}}{x^2 (2n+1)} =$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{4n}}{2n+1}}$$

Correction x^{6n+4}

(b) $$\int x \sin(x^3)\,dx = \int x \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}\,dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n+1)!}\,dx = \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+5}}{(2n+1)!(6n+5)}}$$

(c) $$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n n!} = \sum_{n=0}^{\infty} \frac{(-\frac{2}{3})^n}{n!} = \boxed{e^{-2/3}}$$

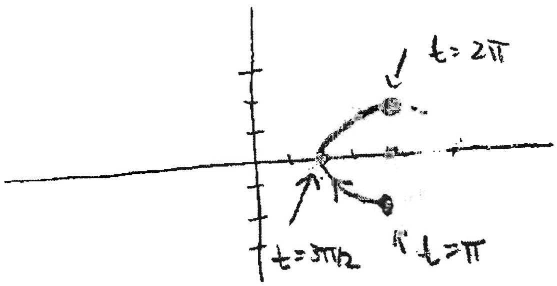
$\boxed{5}$ (10) (a) $x(t) = 2\sin t + 4 \qquad y(t) = 3\cos t \qquad \pi \le t \le 2\pi$

$x = 2\sin t + 4 \qquad y = 3\cos t$

$x - 4 = 2\sin t \qquad \frac{y}{3} = \cos t$

$\frac{x-4}{2} = \sin t$

$\sin^2 t + \cos^2 t = 1$ so $\left(\frac{x-4}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ ellipse

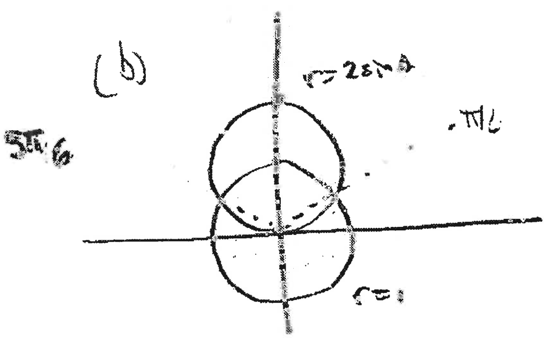


t	$x = 2\sin t + 4$	$y = 3\cos t$
π	4	-3
$3\pi/2$	2	0
2π	4	3

(b) $x'(t) = 2\cos t \quad y'(t) = -3\sin t \qquad L = \int_{\pi}^{2\pi} \sqrt{(2\cos t)^2 + (-3\sin t)^2}\, dt = \int_{\pi}^{2\pi} \sqrt{4\cos^2 t + 9\sin^2 t}\, dt$

$\boxed{6}$ (15) (a) $3 \le r \le 4 \qquad \pi/4 \le \theta \le 3\pi/4$ (see §10.3 7, 9, 11)

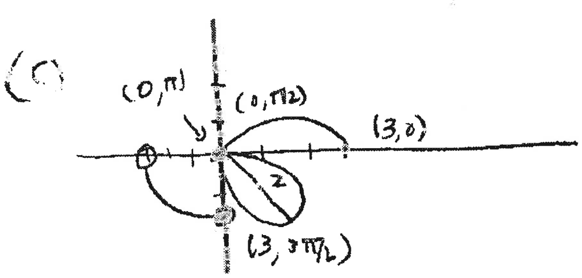
(b)



$2\sin\theta = 1$

$A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/6} 1^2\, d\theta - \frac{1}{2}\int_{0}^{\pi/6} (2\sin\theta)^2\, d\theta\right]$

(c)



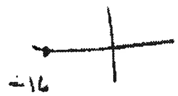
7 (13) (a) $z = -16 \quad r = 16, \theta = \pi$ roots

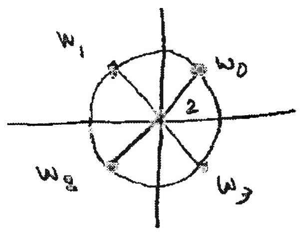
$w_0 = 2 \operatorname{cis} \pi/4 = \sqrt{2} + i\sqrt{2}$

$w_1 = 2 \operatorname{cis} 3\pi/4 = -\sqrt{2} + i\sqrt{2}$

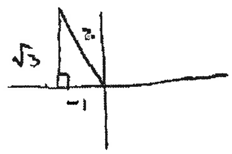
$w_2 = 2 \operatorname{cis} 5\pi/4 = -\sqrt{2} - i\sqrt{2}$

$w_3 = 2 \operatorname{cis} 7\pi/4 = \sqrt{2} - i\sqrt{2}$





(b) $(-1 + i\sqrt{3})^6$



$z = -1 + i\sqrt{3} \Leftrightarrow (-1, \sqrt{3})$

$r = 2, \ \theta = 2\pi/3$

$\cos\theta = -1/2$

$Z = 2 \operatorname{cis} 2\pi/3$

$Z^6 = 2^6 \operatorname{cis} 6 \cdot 2\pi/3 = 2^6 \operatorname{cis} 4\pi$

$= 2^6 (\cos 4\pi + i \sin 4\pi) = 2^6 = \boxed{64}$

(c) $e^{7 + i\pi/6} = e^7 e^{i\pi/6} = e^7 \left[\cos \pi/6 + i \sin \pi/6\right] = e^7 \left[\sqrt{3}/2 + i/2\right] = \boxed{\frac{e^7\sqrt{3}}{2} + \frac{e^7 i}{2}}$

8 (12) a) $T\Sigma = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{7^n \sqrt{n+1}} \cdot \frac{(x+3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x+3)^n}{7^n \sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{n+1}\ (x+3)^n}{7^n}$

b)

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1/x$	$1/2 = 1 \cdot 2^{-1}$
1	$-1/x^2 = -x^{-2}$	$-1/2^2 = -1 \cdot 2^{-2}$
2	$2x^{-3}$	$2 \cdot 2^{-3}$
3	$-2 \cdot 3\ x^{-4}$	$-2 \cdot 3\ 2^{-4}$
4	$2 \cdot 3 \cdot 4 \cdot x^{-5}$	$2 \cdot 3 \cdot 4 \cdot 2^{-5}$
.		
.		
n		$\boxed{(-1)^n n!/2^{n+1}}$

$T\Sigma = \sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(2)}\,(x-2)^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^{n+1}} \cdot \frac{(x-2)^n}{n!}$

$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{2^{n+1}}}$