KEY

Math 34
Calculus II

TUFTS UNIVERSITY
Department of Mathematics
Exam 2

March 31, 2014
All sections

No books, notes, **or calculators**. **TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. You should evaluate standard trigonometric functions like $\tan(\pi/3)$. You are not allowed to quote results about growth rates. You are required to **sign** your exam book. With your signature, you pledge that you have neither given nor received assistance on this exam.

Problem	Point Value	Points
1(a)	10	
1(b)	10	
1(c)	10	
2	10	
3	10	
4	20	
5	12	
6	8	
7	10	
	100	

1. Determine the convergence or divergence of the following series. Justify your answer. State and check hypotheses of any test, rules or theorems you use.(10 points each)

(a) $\sum_{k=1}^{\infty} \frac{\sin^4 k}{5^k}$ positive series $\sim \sum \frac{1}{5^k}$ Convergent geometric series $|r| = \frac{1}{5}$ and $\frac{1}{5} < 1$.

<u>OCT</u> $0 < \frac{\sin^4 k}{5^k} \leq \frac{1}{5^k}$, $k \geq 1$

Since $\sum \frac{1}{5^k}$ is a convergent geometric series with $|r| = |\frac{1}{5}| = \frac{1}{5} < 1$,

ow Σ C's by OCT.

(b) $\sum_{k=1}^{\infty} \frac{2^k}{(k^2+1)^k}$ positive series <u>root</u> test.

$\rho = \lim_{k \to \infty} \left(\frac{2^k}{(k^2+1)^k} \right)^{\frac{1}{k}} = \lim_{k \to \infty} \frac{2}{k^2+1} = 0 < 1$

$\therefore$ Σ C's by the root test.

(c) $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ (positive series) Integral Test

Let $f(x) = \frac{1}{x(\ln x)^2}$, $x \geq 2$

Then ① $f(x)$ is continuous and positive.

② $f(x) = [x(\ln x)^2]^{-1}$ so $f'(x) = -\underbrace{[x(\ln x)^2]^{-2}}_{+}\left[\underbrace{1 \cdot [\ln x]^2}_{+} + \underbrace{x \cdot 2(\ln x) \cdot \frac{1}{x}}_{+}\right]$

$\rightarrow f'(x) < 0$ for $x \geq 2$

So $f(x)$ is decreasing for $x \geq 2$.

$\therefore$ we can use the integral test.

$$\text{Int} = \int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \int_2^t \frac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \left[-\frac{1}{\ln x} \Big|_2^t \right]$$

$$= \lim_{t \to \infty} \left[\underbrace{-\frac{1}{\ln t}}_{\to 0} + \frac{1}{\ln 2} \right] = \frac{1}{\ln 2}$$

$$\left[\begin{array}{l} \text{Form } \int \frac{du}{u^2} = -\frac{1}{u} + C \\ (\text{with } u = \ln x) = -\frac{1}{\ln x} + C \end{array} \right]$$

Conclusion: Since Int c's to $\frac{1}{\ln 2}$, the series converges by the integral test.

2. (8 points) Determine the convergence or divergence of the following series. Justify your answer. State and check hypotheses of any test, rules or theorems you use. *If the series converges, find its sum.*

$$\sum_{k=1}^{\infty} \frac{(-3)^k}{5^{k+1}} = -\frac{3}{5^2} + \frac{(-3)^2}{5^3} + \frac{(-3)^3}{5^4} + \cdots$$

geometric series. $|r| = \left|\frac{-3}{5}\right| = \frac{3}{5} < 1$ so the series

C's to the sum $S = \frac{a}{1-r} = \frac{-3/5^2}{1-(-3/5)} = \frac{-3/5^2}{8/5}$

$= -\frac{3}{5^2} \cdot \frac{5}{8} = \frac{-3}{40}$

3. (10 points) Quadratic Approximation

(a) Find the 2nd-order Taylor polynomial $p_2(x)$ for $f(x) = 2x + \cos x$ centered at 0.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$2x + \cos x$	$0 + \cos 0 = 1$
1	$2 - \sin x$	$2 - \sin 0 = 2 - 0 = 2$
2	$-\cos x$	$-\cos 0 = -1$

$p_2(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!} x^2$

$\boxed{p_2(x) = 1 + 2x - \frac{1}{2}x^2}$

(b) Use the polynomial you found in part (a) to approximate $f(1/2)$. Simplify your answer.

$f(\frac{1}{2}) \approx p_2(\frac{1}{2}) = 1 + 2(\frac{1}{2}) - \frac{1}{2}(\frac{1}{2})^2$

$= 1 + 1 - \frac{1}{8} = 2 - \frac{1}{8} = \frac{15}{8}$

4. (20 points) Determine whether the following series converge absolutely, conditionally, or diverge. Justify your answer. State and check hypotheses of any test, rules or theorems you use.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k \cdot \pi}{tan^{-1}(k)}$

① AbC? $\sum_{k=1}^{\infty} \frac{\pi}{\tan^{-1} k}$

DN test $\lim_{k\to\infty} \frac{\pi}{\tan^{-1} k} = \pi/\pi/2 = 2$

$\therefore \lim_{k\to\infty} \frac{(-1)^k \pi}{\tan^{-1} \pi} \neq 0$ (oscillates)

and this series diverges by the divergence test

DONE

(b) $\sum_{k=1}^{\infty} \frac{(-6)^k}{k!}$

① AbC? $\sum_{k=1}^{\infty} \frac{6^k}{k!}$

RATIO TEST (RATIFACE)

$r = \lim_{k\to\infty} \left[\frac{6^{k+1}}{(k+1)!} \cdot \frac{k!}{6^k} \right]$

$= \lim_{k\to\infty} \left[\frac{6 \cdot 6^k}{6^k} \cdot \frac{k!}{(k+1)\cdot k!} \right] = \lim_{k\to\infty} \left[\frac{6}{k+1} \right] = 0 < 1$

$\therefore$ This series is AbC by RATIFACE.

RoC IoC

5. (12 points) Find the radius of convergence and interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cdot (x-3)^k}{k^2 \cdot 2^k}.$$

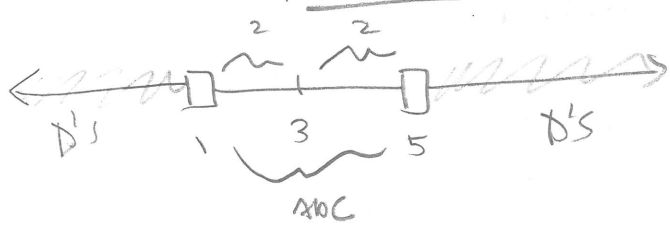
center $a=3$

$\boxed{1}$ RATFACE $r = \lim_{k\to\infty} \left| \frac{(x-3)^{k+1}}{(k+1)^2\, 2^{k+1}} \cdot \frac{k^2 2^k}{(x-3)^k} \right| = \frac{|x-3|}{2} \lim_{k\to\infty} \frac{k^2}{(k+1)^2}$

$= \frac{|x-3|}{2} \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2 = \frac{|x-3|}{2} \lim_{k\to\infty} \left(\frac{1}{1+\underbrace{1/k}_{\to 0}}\right)^2 = \frac{|x-3|}{2}$

$\boxed{2}$ $r<1$ $\quad \frac{|x-3|}{2} < 1 \Rightarrow |x-3| < 2 \qquad -2 < x-3 < 2$

$\boxed{\text{RoC} = 2} \qquad 1 < x < 5$



$\boxed{3}$ endpts (i) $x=1$ $\quad \sum_{k=1}^{\infty} \frac{(-1)^k(-2)^k}{k^2\, 2^k} = \sum_{k=1}^{\infty} \frac{2^k}{k^2 2^k} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ conv $p\text{-}\Sigma$, $p=2>1$

(ii) $x=5$ $\quad \sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{2^k k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$

<u>option 1</u> This series is AbC. $\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k^2}\right| = \sum \frac{1}{k^2}$ conv $p\text{-}\Sigma$ $p=2>1$

or

<u>option 2</u> A.S. Test ① $\lim_{k\to\infty} \frac{1}{k^2} = 0$ ✓

② $\frac{1}{k^2} > 0$, $k \geq 1$ ✓

③ $\frac{1}{(k+1)^2} \leq \frac{1}{k^2}$, $k \geq 1$ ✓

Σ passes A.S.T + hence C's

$\boxed{\text{IoC} = [1,5]}$

6. (8 points) Given that $f(x) = \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$ for $-1 \le x < 1$, find a power series representation (centered at 0) for $g(x) = x^5 \ln(1 + \frac{x}{3})$ and give the $\underbrace{\text{interval of convergence}}_{\text{IoC}}$.

$$g(x) = x^5 \ln\left(1 - \left(-\frac{x}{3}\right)\right) = -x^5 \sum_{k=1}^{\infty} \frac{(-x/3)^k}{k}$$

$$= -x^5 \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{3^k \cdot k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k+5}}{3^k \cdot k}$$

for $-1 \le (-x/3) < 1$

$-3 \le -x < 3$

$3 \ge x > -3$

or

$$\boxed{\text{IoC} \quad -3 < x \le 3}$$

7. (10 pts) **True/false questions.** Indicate your answer by circling T or F.

(a) The series $\sum_{k=1}^{\infty} 1^k$ converges to 1. T F

F D'S by Div test $\lim_{k\to\infty} 1 > 1$ not 0.

(b) The antiderivative of the series $\sum_{k=0}^{\infty}(-2)^k x^k$ is $C + \sum_{k=0}^{\infty} \frac{(-2)^{k+1}x^{k+1}}{k+1}$ T F

F: $C + \sum_{k=0}^{\infty} \frac{(-2)^k x^{k+1}}{k+1}$

(c) If the power series $\sum_{k=0}^{\infty} c_k(x-2)^n$ has radius of convergence 3 then the series diverges at $x = 4$. T F

$|x-2| < 3$ F Σ c's at $x=4$

$-3 < x-2 < 3$

$-1 < x < 5$

(d) If $\sum_{k=1}^{\infty} a_k$ converges then $\sum_{k=1}^{\infty} |a_k|$ converges. T F

F If $\Sigma |a_k|$ c's then Σa_k c's

(e) If the sequence $\{a_1 + a_2 + \cdots + a_n\}$ converges to S then $\sum_{k=1}^{\infty} a_k$ converges to S. T F

T - d-fn of series convergence

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Name _____

Circle your section:

32-01 Hao Liang TThF 8:30-9:20

32-02 Mary Glaser TThF 12-12:50

32-03 Gail Kaufmann TTHF 12-12:50

32-04 Thomas Benson TTh1:30-2:20,F2:30-3:20

I pledge that I have neither given nor received assistance on this exam.

Signature _____