SOLUTIONS

Math 34
Calculus II

TUFTS UNIVERSITY
Department of Mathematics
Exam 2

November 3, 2014
All sections

No books, notes, **or calculators. TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. You should evaluate standard trigonometric functions like $\tan(\pi/3)$. You are not allowed to quote results about growth rates. You are required to **sign** your exam on the last page. With your signature, you pledge that you have neither given nor received assistance on this exam. On the inside of the last page there is a blank side of paper for scratch work that is not to be graded.

Problem	Point Value	Points
1(a)	10	
1(b)	10	
2 (a)	10	
2 (b)	10	
3	12	
4	12	
5 (a)	11	
5 (b)	11	
6	14	
	100	

1. (20 points) Determine the convergence or divergence of the following series. Justify your answer. State and check hypotheses of any test, rules or theorems you use. (10 points each)

1a) $\sum_{k=1}^{\infty} \frac{5+\cos(k)}{\sqrt{k}}$

$$\frac{4}{\sqrt{k}} \leq \frac{5+\cos(k)}{\sqrt{k}} \leq \frac{6}{\sqrt{k}}$$

$\sum_{k=1}^{\infty} \frac{4}{\sqrt{k}}$ is a divergent p-series with

$p = \frac{1}{2} \not> 1$.

By DCT, $\sum_{k=1}^{\infty} \frac{5+\cos(k)}{\sqrt{k}}$ DIVERGES

Determine the convergence or divergence of the following series. Justify your answer. State and check hypotheses of any test, rules or theorems you use.

1b) $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$ Integral Test

$f(k) = \frac{1}{k\sqrt{\ln k}}$. Let $f(x) = \frac{1}{x\sqrt{\ln x}}$ for $x \geq 2$.

Check conditions:

1) No asymptotes, so $f(x)$ is continuous.

2) For $x \geq 2$, $\frac{1}{\underbrace{x}_{(+)}\underbrace{\sqrt{\ln x}}_{(+)}}$. So $f(x)$ is positive.

3) $f(x) = \frac{1}{x\sqrt{\ln x}} = [x(\ln x)^{1/2}]^{-1}$.

$f'(x) = -[x(\ln x)^{1/2}]^{-2}\left[(\ln x)^{1/2} + x\left(\frac{1}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right)\right]$

$= \underbrace{\frac{-1}{(x\sqrt{\ln x})^2}}_{(-)}\left[\underbrace{\sqrt{\ln x}}_{(+)} + \underbrace{\frac{1}{2\sqrt{\ln x}}}_{(+)}\right]$ for $x \geq 2$

$f'(x) < 0$ for $x \geq 2$, so $f(x)$ is decreasing for $x \geq 2$.

We can use the Integral Test!

$$\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{\ln x}} \qquad u = \ln x, \quad du = \frac{dx}{x}$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{\sqrt{u}}$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-1/2}\, du$$

$$= \lim_{b\to\infty} 2u^{1/2}\Big|_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \left[2\underbrace{(\ln b)^{1/2}}_{\downarrow} - 2(\ln 2)^{1/2}\right]$$

$(\ln b)^{1/2} \to +\infty$ as $b \to \infty$

$= +\infty$. The integral diverges, so $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$ diverges by the Integral Test.

2. (20 points) Determine the convergence or divergence of the following series. Justify your answer. State and check hypotheses of any test, rules or theorems you use. *If the series converges, find its sum.* (10 points each)

2a) $\sum_{k=1}^{\infty} \frac{(-5)^{k+1}}{3^{2k}} = \sum_{k=1}^{\infty} \frac{(-5)^k(-5)}{(3^2)^k} = -5\sum_{k=1}^{\infty}\left(\frac{-5}{9}\right)^k$

This is a geometric series with

$|r| = \left|\frac{-5}{9}\right| = \frac{5}{9} < 1$, so the series converges.

It converges to

$$S = \frac{-5\left(\frac{-5}{9}\right)^1}{1-\left(\frac{-5}{9}\right)} = \frac{\frac{25}{9}}{\frac{14}{9}}$$

$$= \frac{25}{9} \cdot \frac{9}{14}$$

$$= \boxed{\frac{25}{14}}$$

Determine the convergence or divergence of the following series. Justify your answer. State and check hypotheses of any test, rules or theorems you use. *If the series converges, find its sum.*

2b) $\sum_{k=1}^{\infty}\left(\sin\left(\frac{\pi}{k+1}\right)-\sin\left(\frac{\pi}{k+2}\right)\right)$ Telescoping Series

$$= \lim_{n\to\infty}\left[\left(\sin\left(\frac{\pi}{1+1}\right)-\sin\left(\frac{\pi}{1+2}\right)\right)+\left(\sin\left(\frac{\pi}{2+1}\right)-\sin\left(\frac{\pi}{2+2}\right)\right)+\cdots+\left(\sin\left(\frac{\pi}{n+1}\right)-\sin\left(\frac{\pi}{n+2}\right)\right)\right]$$

rearrange parentheses

$$= \lim_{n\to\infty}\left[\sin\left(\frac{\pi}{1+1}\right)\left(-\sin\left(\frac{\pi}{1+2}\right)+\sin\left(\frac{\pi}{2+1}\right)\right)+\left(-\sin\left(\frac{\pi}{2+2}\right)+\cdots+\left(-\sin\left(\frac{\pi}{n+1}\right)+\sin\left(\frac{\pi}{n+1}\right)\right)-\sin\left(\frac{\pi}{n+2}\right)\right]$$

$$= \lim_{n\to\infty}\left(\sin\left(\frac{\pi}{2}\right)-\sin\left(\frac{\pi}{n+2}\right)\right)$$

$$= \lim_{n\to\infty}\left(1-\sin\left(\frac{\pi}{n+2}\right)\right)$$

$\sin\left(\frac{\pi}{n+2}\right)\to 0$ as $n\to\infty$

$$= \boxed{1}$$

The series converges to 1.

3. (12 points) Quadratic Approximation

(a) Find the 2nd-order Taylor polynomial $p_2(x)$ for $f(x) = \frac{x}{e} + \ln x$ centered at e.

n	$f^{(n)}(x)$	$f^{(n)}(e)$
0	$\frac{x}{e} + \ln x$	$\frac{e}{e} + \ln(e) = 1+1 = \boxed{2}$
1	$\frac{1}{e} + \frac{1}{x}$	$\frac{1}{e} + \frac{1}{e} = \boxed{\frac{2}{e}}$
2	$\frac{-1}{x^2}$	$\boxed{\frac{-1}{e^2}}$

$$p_2(x) = f(e) + f'(e)(x-e) + \frac{f''(e)}{2!}(x-e)^2$$

$$= 2 + \frac{2}{e}(x-e) + \frac{\left(\frac{-1}{e^2}\right)}{2}(x-e)^2$$

$$\boxed{p_2(x) = 2 + \frac{2(x-e)}{e} - \frac{(x-e)^2}{2e^2}}$$

(b) Use the polynomial you found in part (a) to approximate $f(e/2)$.

$$f\left(\frac{e}{2}\right) \approx p_2\left(\frac{e}{2}\right) = 2 + \frac{2\left(\frac{e}{2} - e\right)}{e} - \frac{\left(\frac{e}{2} - e\right)^2}{2e^2}$$

$$= 2 + \frac{\left(\frac{2e}{2} - 2e\right)}{e} - \frac{\left(\frac{-e}{2}\right)^2}{2e^2}$$

$$= 2 + \frac{(-e)}{e} - \left(\frac{e^2}{4}\right)\left(\frac{1}{2e^2}\right)$$

$$= 2 - 1 - \frac{1}{8}$$

$$= 1 - \frac{1}{8}$$

$$= \boxed{\frac{7}{8}}$$

4. (12 points)

(a) What is the conclusion of the Root Test for the series $\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n$?

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n-1}{n}\right)^n} = \lim_{n\to\infty} \left(\frac{n-1}{n}\right)$$

$$= 1 \quad \text{INCONCLUSIVE}$$

(b) Show that $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1}$.

$$\left(1-\tfrac{1}{n}\right)^n = e^{\ln\left(1-\frac{1}{n}\right)^n} = e^{n\ln\left(1-\frac{1}{n}\right)}$$

$$L = \lim_{n\to\infty} n\ln\left(1-\tfrac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{1/x} \quad \text{(switch to continuous variable } x\text{)}$$

$$\overset{\text{L'H}}{\underset{0/0}{=}} \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{x}}\left(\frac{1}{x^2}\right)}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{-1}{1-\frac{1}{x}} = -1$$

$$\lim_{n\to\infty} \left(1-\tfrac{1}{n}\right)^n = e^{L} = e^{-1}$$

(c) Now use part b) above to help you determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n$.

$$\lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty} \left(1-\tfrac{1}{n}\right)^n = e^{-1} \neq 0$$

By the Divergence Test, the series Diverges.

5. (22 points) Determine whether each of the following series converges absolutely, conditionally, or diverges. Justify your answer. State and check hypotheses of any test, rules or theorems you use. (11 points each)

(a) $\sum_{k=1}^{\infty}(-1)^k \frac{k^2}{k!}$

RATFACE:

$$\sum_{k=1}^{\infty}\left|\frac{(-1)^k k^2}{k!}\right| = \sum_{k=1}^{\infty}\frac{k^2}{k!}$$

$$r = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{(k+1)^2}{(k+1)!}\cdot\frac{k!}{k^2}$$

$$= \lim_{k\to\infty}\frac{(k+1)^2}{(k+1)\,k^2}$$

$$= \lim_{k\to\infty}\frac{k+1}{k^2}$$

$$= \lim_{k\to\infty}\frac{1+(1/k)\to 0}{(k)\to+\infty}$$

$$= 0$$

Since $0 \le r < 1$, $\sum_{k=1}^{\infty}\left|\frac{(-1)^k k^2}{k!}\right|$ converges, so the original series $\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{k!}$

Converges absolutely

5(b) Determine whether the following series converges absolutely, conditionally, or diverges.

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt[3]{k-1}}$$

Check AbC:

$$\sum_{k=2}^{\infty} \left| \frac{(-1)^k}{\sqrt[3]{k-1}} \right| = \sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k-1}}$$

The terms of $\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k-1}}$ and $\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}}$ are positive,

and $L = \lim_{k\to\infty} \frac{1}{\sqrt[3]{k-1}} \cdot \frac{\sqrt[3]{k}}{1}$

$$= \lim_{k\to\infty} \left(\frac{k}{k-1}\right)^{1/3} = (1)^{1/3} = 1$$

Since $0 < 1 < \infty$, we can use the LCT.

$\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}}$ is a divergent p-series with $p = \frac{1}{3} \neq 1$,

so by the LCT, $\sum_{k=2}^{\infty} \left| \frac{(-1)^k}{\sqrt[3]{k-1}} \right|$ diverges, and

$\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt[3]{k-1}}$ does <u>not</u> converge absolutely.

Now check with Alternating Series Test:

$\sum_{k=2}^{\infty} (-1)^k \underbrace{\frac{1}{\sqrt[3]{k-1}}}_{>0}$

1) $\lim_{k\to\infty} \frac{1}{\sqrt[3]{k-1}} = 0$ ✓

2) As $k \to \infty$, $\sqrt[3]{k-1} \uparrow \infty$, so $\frac{1}{\sqrt[3]{k-1}} \downarrow 0$.

Decreasing terms. ✓

The series converges by the Alternating Series Test,

so $\sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt[3]{k-1}}$ is <u>Conditionally Convergent</u>.

6. (14 points) Find the radius of convergence and interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k \cdot 9^k}.$$

RATFACE: $r = \lim_{k\to\infty} \left| \frac{(x-2)^{(k+1)}}{(k+1)\,9^{(k+1)}} \cdot \frac{k\,9^k}{(x-2)^k} \right|$

$$= \lim_{k\to\infty} \left| \frac{(x-2)\,k}{9(k+1)} \right|$$

$$= \frac{|x-2|}{9} \lim_{k\to\infty} \frac{k}{k+1}$$

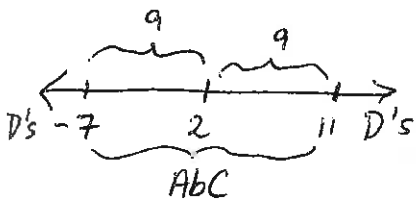
$$= \frac{|x-2|}{9}$$

For AbC, $|r| < 1$, so we want $\frac{|x-2|}{9} < 1$.

$$\frac{|x-2|}{9} < 1 \implies |x-2| < 9$$

$$\implies -9 < x-2 < 9$$

$$\implies -7 < x < 11$$



RoC = 9

Check Endpoints:

$x = -7$: $\sum_{k=1}^{\infty} \frac{(-9)^k}{k\,9^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ Converges (alternating harmonic)

$x = 11$: $\sum_{k=1}^{\infty} \frac{9^k}{k\,9^k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic series)

IoC: $[-7, 11)$