Math 34

Tufts University
Department of Mathematics
Exam 2

November 18, 2013
12:00-1:20

No books, notes or calculators are allowed. Cross out what you do not want us to grade. You must show all your work in order to receive full credit unless otherwise indicated. You may **not** refer to growth rates when taking limits. You are expected to evaluate all trigonometric values. Please write neatly. If you use the scratch work page at the end of the exam for work you want graded, please make sure you indicate that. You are required to sign your exam. With your signature, you pledge that you have neither given nor received assistance on this exam.

KEY

Problem	Point Value	Points
1	12	
2	10	
3	15	
4	12	
5	6	
6	5	
7	10	
8	10	
9	12	
10	8	
	100	

1. (12 points) Determine whether each of the following sequences converges or diverges. If the sequence converges, find its limit. Justify your answers. You may not quote any results about growth rates.

(a) $a_n = \dfrac{n}{\ln n}$

$$\lim_{n\to\infty} \frac{\overset{\to\infty}{n}}{\underset{\to\infty}{\ln n}} = \lim_{x\to\infty} \frac{\overset{\to\infty}{x}}{\underset{\to\infty}{\ln x}} \overset{(L)}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} x \to \infty$$

sequence diverges

(b) $a_n = \dfrac{(-1)^n + 2}{7n - 1}$

Squeeze

$$\frac{1}{7n-1} \le \frac{(-1)^n+2}{7n-1} \le \frac{3}{7n-1}$$

$\to 0$

Sequence converges to 0.

(c) $a_n = \dfrac{n}{\arctan n}$

$$\lim_{n\to\infty} \frac{\overset{\to\infty}{n}}{\underset{\to\pi/2}{\arctan n}} \to \infty$$

sequence diverges

2. (10 points)

Consider the series $\sum_{n=1}^{\infty}\left(\frac{4}{n}-\frac{4}{n+1}\right)$

(a) Write down *but do not simplify* an expression for S_n, the sum of the first n terms of the series.

$$S_n = \left(4-\frac{4}{2}\right)+\left(\frac{4}{2}-\frac{4}{3}\right)+\left(\frac{4}{3}-\frac{4}{4}\right)+\cdots+\left(\frac{4}{n}-\frac{4}{n+1}\right)$$

(b) Simplify the expression you wrote down in part (a) above.

$$S_n = \left(4+\cancel{\frac{4}{2}}+\cancel{\frac{4}{3}}+\cdots+\cancel{\frac{4}{n}}\right)-\left(\cancel{\frac{4}{2}}+\cancel{\frac{4}{3}}+\cdots+\cancel{\frac{4}{n}}+\frac{4}{n+1}\right)$$

$$S_n = 4-\frac{4}{n+1}$$

(c) Based on your work in part (b) above, determine whether the series converges or diverges. If it converges find its sum.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(4-\underbrace{\frac{4}{n+1}}_{\downarrow\, 0}\right)=4$$

$\therefore$ The series converges to the sum $S=4$.

3. (15 points) Determine whether each of following series is absolutely convergent (AbC), conditionally convergent (CC) or divergent (D). *No justification needed.* No partial credit. Fill in the blank after each series with AbC, CC, or D.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ AbC

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ CC

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \sin k$ D

(d) $\displaystyle\sum_{k=1}^{\infty} 1^k$ D

(e) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{e^k}{\pi^k}$ AbC

4. (12 points) Find the second-order Taylor polynomial $p_2(x)$ for each of the following functions.

(a) $f(x) = e^x \sin x + 3$, centered at $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$e^x \sin x + 3$	3
1	$e^x \sin x + e^x \cos x$	$0+1=1$
2	$\cancel{e^x \sin x} + e^x \cos x + e^x \cos x - \cancel{e^x \sin x}$	2

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$p_2(x) = 3 + x + \frac{2}{2!}x^2 = 3 + x + x^2$$

(b) $g(x) = 4\ln x$ centered at $a = 2$.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$4\ln x$	$4\ln 2$ (or $\ln 16$)
1	$\frac{4}{x}$	2
2	$-\frac{4}{x^2}$	-1

$$p_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$$

$$p_2(x) = 4\ln 2 + 2(x-2) + \frac{-1}{2!}(x-2)^2$$

$$p_2(x) = 4\ln 2 + 2(x-2) - \frac{1}{2}(x-2)^2$$

5. (6 points) The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{3 \cdot 10^n}$ passes the alternating series test.

Find the sum of the series to within an accuracy of $\frac{1}{2500}$. Simplify your answer.

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{3 \cdot 10^n} = \frac{1}{3} - \frac{1}{3 \cdot 10} + \frac{1}{3 \cdot 10^2} - \frac{1}{3 \cdot 10^3} + \cdots$$

$$\frac{1}{300} \qquad \frac{1}{3000} < \frac{1}{2500}$$

$$|S - S_3| \le \frac{1}{3000} < \frac{1}{2500}$$

$$S_3 = \frac{1}{3} - \frac{1}{30} + \frac{1}{300} = \frac{100 - 10 + 1}{300} = \frac{91}{300}$$

6. (5 points) Determine whether the following series converges or diverges. JUSTIFY YOUR ANSWER. STATE AND CHECK HYPOTHESES OF ANY TEST, RULES OR THEOREM YOU USE.

$$\sum_{n=1}^{\infty} \frac{10n^5 + 20n^4}{n^7 + n + 1} \sim \sum_{n=1}^{\infty} \frac{n^5}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

positive Σ

LCT $\quad L = \lim_{n\to\infty} \frac{10n^5 + 20n^4}{n^7 + n + 1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{10n^7 + 20n^6}{n^7 + n + 1}$

$$= \lim_{n\to\infty} \frac{10 + \underbrace{20/n}_{\to 0}}{1 + \underbrace{1/n^6}_{0} + \underbrace{1/n^7}_{0}} = 10$$

Since $0 < 10 < +\infty$ and since $\sum \frac{1}{n^2}$ is a convergent p-Σ $(p = 2 > 1)$, our series converges.

7. (10 points) Find the radius of convergence and interval of convergence of the following power series. Justify your answer.

$$\sum_{k=1}^{\infty}(-1)^k \frac{(x-1)^k}{k \cdot 6^k} \qquad \text{Center } a = 1$$

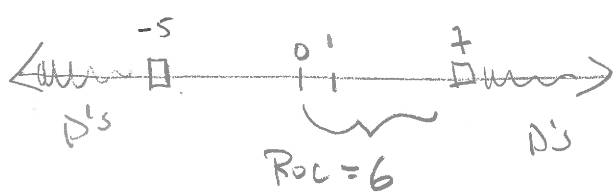
$\boxed{1}$ $\underline{\text{RATTACE}}$ $\quad r = \lim_{k\to\infty} \left| \frac{(x-1)^{k+1}}{(k+1)6^{k+1}} \frac{k \cdot 6^k}{(x-1)^k} \right|$

$= \lim_{k\to\infty} \left| \frac{(x-1)}{6} \cdot \frac{k}{k+1} \right| = \frac{|x-1|}{6} \lim_{k\to\infty} \frac{k}{k+1} = \frac{|x-1|}{6} \lim_{k\to\infty} \frac{1}{1+(1/k)}$

$= \frac{|x-1|}{6}$

$\boxed{2}$ Solve $r < 1$ $\quad \frac{|x-1|}{6} < 1 \Leftrightarrow |x-1| < 6 \Leftrightarrow -6 < x-1 < 6$

$-5 < x < 7$



$\boxed{3}$ Endpts (i) $x = -5$

$$\sum_{k=1}^{\infty} \frac{(-1)^k(-5-1)^k}{k \cdot 6^k} = \sum_{k=1}^{\infty} \frac{(-1)^k(-6)^k}{k \cdot 6^k} = \sum_{k=1}^{\infty} \frac{6^k}{k 6^k} = \sum_{k=1}^{\infty} \frac{1}{k} \text{ divergent}$$

harmonic series

(ii) $x = 7$

$$\sum_{k=1}^{\infty} \frac{(-1)^k(7-1)^k}{k 6^k} = \sum_{k=1}^{\infty} \frac{(-1)^k 6^k}{k \cdot 6^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

alternating harmonic series - converges:

A.S.T ① $\lim_{k\to\infty} \frac{1}{k} = 0$

② $\frac{1}{k} > 0$, $k \geq 1$

③ $\frac{1}{k+1} \leq \frac{1}{k}$ for all k

radius of convergence $= 6$

interval of convergence : $(-5, 7]$

8. (10 points) Use the *definition of the Taylor series* to find the Taylor series for $f(x) = \dfrac{1}{(x+2)^2}$ centered at $a = 1$. Write your answer in summation notation and simplify your answer. You *do not* need to find the radius of convergence or interval of convergence.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\frac{1}{(x+2)^2}$	$\frac{1}{3^2}$
1	$-2(x+2)^{-3}$	$-2/3^3$
2	$2\cdot 3(x+2)^{-4}$	$2\cdot 3/3^4$
3	$-2\cdot 3\cdot 4(x+1)^{-5}$	$-2\cdot 3\cdot 4/3^5$
⋮		
n		$\frac{(-1)^n (n+1)!}{3^{n+2}}$

$$T\Sigma = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!\,(x-1)^n}{3^{n+2}\,n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)n!\,(x-1)^n}{3^{n+2}\cdot n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-1)^n}{3^{n+2}}$$

9. (12 points) Given that $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for all x with $|x| < 1$, find a series representation for each of the following functions along with the radius of convergence.. You do *not* need to find the interval of convergence. Show all work.

(a) $g(x) = \frac{1}{1-4x^2}$ $= \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$

$|4x^2| < 1$

$4|x|^2 < 1$

$|x|^2 < \frac{1}{4}$

$|x| < \frac{1}{2}$

$-\frac{1}{2} < x < \frac{1}{2}$

radius of convergence $= \frac{1}{2}$

(b) $j(x) = -\ln(1-x)$

$-\ln(1-x) = C + \int \frac{dx}{1-x} = C + \int \sum_{n=0}^{\infty} x^n dx$

$-\ln(1-x) = C + \sum_{n=0}^{\infty} \int x^n dx$

$-\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

Solve for C: If $x=0$ $\quad -\ln(1) = C + 0$

$0 = C + 0 \Rightarrow C = 0$

$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

radius of convergence is the same as the radius of convergence for $\frac{1}{1-x}$, which is 1.

10. (8 points) Find the function represented by the following power series and its interval of convergence: $\sum_{n=0}^{\infty} \left(\frac{x}{5} - 2\right)^n$

This is a geometric series.

If $\left|\frac{x}{5}-2\right| < 1$, it converges to $f(x) = \frac{1}{1-\left(\frac{x}{5}-2\right)}$

$-1 < \frac{x}{5} - 2 < 1$

$1 < \frac{x}{5} < 3$

$\boxed{5 < x < 15}$

interval of convergence

$f(x) = \frac{1}{1-\frac{x}{5}+2} = \frac{1}{3-\frac{x}{5}}$

$\boxed{f(x) = \frac{5}{15-x}}$