Math 12 Exam II Solutions Fall 2010

[1] T T F F F F T

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[2] a) $a_n = (-1)^n\left(\frac{14n^2+11}{2n^2+3}\right)$

$$\lim_{n\to\infty}\frac{14n^2+11}{2n^2+3} = \lim_{n\to\infty}\frac{14+11/n^2}{2+3/n^2} = 7$$

$\therefore \lim_{n\to\infty}(-1)^n\frac{14n^2+11}{2n^2+3}$ d.n.e. (does not exist)

oscillates: even terms $\to 7$, odd terms $\to -7$

Sequence diverges

b) $a_n = \frac{(2(n+1))!}{n^2\,(2n)!}$

$$\lim_{n\to\infty}\frac{(2(n+1))!}{n^2(2n)!} = \lim_{n\to\infty}\frac{(2n+2)!}{n^2(2n)!}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)(2n)!}{n^2\,(2n)!} = \lim_{n\to\infty}\frac{4n^2+6n+2}{n^2}$$

$$= \lim_{n\to\infty} 4+\underset{\to 0}{\frac{6}{n}}+\underset{\to 0}{\frac{2}{n^2}} = 4$$ sequence converges to 4

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[3] a) $\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right)$

$$S_n = \left(\cos\frac{\pi}{1}-\cos\frac{\pi}{2}\right)+\left(\cos\frac{\pi}{2}-\cos\frac{\pi}{3}\right)+\cdots+\left(\cos\frac{\pi}{n}-\cos\frac{\pi}{(n+1)}\right)$$

$$=\left(\cos\pi+\cos\pi/2+\cos\pi/3+\cdots+\cos\pi/n\right)-\left(\cos\pi/2+\cos\pi/3+\cdots+\cos\pi/n+\cos\pi/(n+1)\right)$$

$$=\cos\pi-\cos\left(\pi/(n+1)\right)$$

b) $\lim_{n\to\infty}\cos\pi-\cos\left(\underset{\to 0}{\pi/(n+1)}\right) = \cos\pi-\cos 0 = -1-1 = -2$ Series C's to $S=-2$

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[4] a) $\sum_{n=1}^{\infty}\frac{3^{n+1}}{\pi^{n-1}} = 3^2+\frac{3^3}{\pi}+\frac{3^4}{\pi^2}+\frac{3^5}{\pi^3}+\cdots$ Geometric $|r| = \left|\frac{3}{\pi}\right|<1$

Σ C's to $S = \frac{a}{1-r} = \frac{9}{1-3/\pi} = \frac{9\pi}{\pi-3}$

b) $\sum_{n=1}^{\infty}\frac{e^n}{n^2}$ Div. Test

$$\lim_{n\to\infty}\frac{e^n}{n^2}\begin{matrix}\to\infty\\ \to\infty\end{matrix} = \lim_{x\to\infty}\frac{e^x}{x^2}\begin{matrix}\to\infty\\ \to\infty\end{matrix}$$

$$\overset{L}{=}\lim_{x\to\infty}\frac{e^x}{2x}\begin{matrix}\to\infty\\ \to\infty\end{matrix} \overset{L}{=}\lim_{x\to\infty}\frac{e^x}{2}\to\infty$$ Σ D's by Div. Test

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[5] $\sum_{n=1}^{\infty}(-1)^n\frac{\arctan n}{n^2+n}$ AbC? $\sum_{n=1}^{\infty}\frac{\arctan n}{n^2+n}$

OCT. $0\le\frac{\arctan n}{n^2+n}<\frac{\pi/2}{n^2+n}<\frac{\pi/2}{n^2}$, $n\ge 1$

Since $\pi/2\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a non-zero multiple of a convergent p-Σ $(p=2>1)$

our Series is AbC

6 $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$ DN. Test $\lim_{n\to\infty} \frac{n \to \infty}{\ln n \to \infty} = \lim_{x\to\infty} \frac{x \to \infty}{\ln x \to \infty} \overset{L}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x \Rightarrow \infty$

$\therefore \lim_{n\to\infty} (-1)^n \frac{n}{\ln n} \neq 0$ (oscillates) and Σ diverges by DN. Test.

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7 a) $\sum_{n=1}^{\infty} (-1)^n \frac{(x+4)^n}{5^n \cdot n^2}$ RATFACE: $\rho = \lim_{n\to\infty} \left| \frac{(x+4)^{n+1}}{5^{n+1}(n+1)^2} \cdot \frac{5^n n^2}{(x+4)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+4)^{n+1} \cdot 5^n \cdot n^2}{(x+4)^n \cdot 5^{n+1} (n+1)^2} \right|$

$= \frac{|x+4|}{5} \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \frac{|x+4|}{5} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \frac{|x+4|}{5} \lim_{n\to\infty} \left(\frac{1}{1+(\underbrace{1/n}_{\to 0})}\right)^2 \to \frac{|x+4|}{5}$

$\frac{|x+4|}{5} < 1 \iff |x+4| < 5$ ROC = 5

b) $|x+4| < 5$ $-5 < x+4 < 5$ $-9 < x < 1$

$x = -9$ $\sum_{n=1}^{\infty} \frac{(-1)^n (-9+4)^n}{5^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n(-5)^n}{5^n n^2} = \sum_{n=1}^{\infty} \frac{5^n}{5^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$x = 1$ $\sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{5^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

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8 a) $\frac{2x^2}{4+3x} = 2x^2\left(\frac{1}{4+3x}\right) = \frac{2x^2}{4}\left(\frac{1}{1+\frac{3x}{4}}\right) = \frac{x^2}{2}\left(\frac{1}{1-(-3x/4)}\right) = \frac{x^2}{2}\sum_{n=0}^{\infty}\left(\frac{-3x}{4}\right)^n$

$\left(\left|\frac{3x}{4}\right| < 1 \iff |x| < 4/3 \quad \text{ROC} = 4/3\right) = \frac{x^2}{2}\sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{n+2}}{2\cdot 4^n}$

b) $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$, $|x| < 1$

$\ln(1+x) = C + \int \frac{dx}{1+x} = C + \int \sum_{n=0}^{\infty} (-1)^n x^n = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

$\ln(1+x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ set $x=0$ $\ln(1) = 0 = C + 0 \to C = 0$

$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$, ROC = 1

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9 $\sum_{n=1}^{\infty} (-1)^n \frac{n+6}{n^4+n}$

(a) AST. ① $\lim_{n\to\infty} \frac{n+6}{n^4+n} = \lim_{n\to\infty} \frac{\overbrace{1/n^3 + 6/n^4}^{\to 0}}{1+\underbrace{(1/n^3)}_{\to 0}} = \frac{0}{1} = 0$ ✓

② $\frac{n+6}{n^4+n} > 0$

③ claims $\left\{\frac{n+6}{n^4+n}\right\}$ decreases $(4x^4+24x^3+x+6)$

Let $f(x) = \frac{x+6}{x^4+x}$, $x \geq 1$. $f'(x) = \frac{(x^4+x)-(x+6)(4x^3+1)}{(x^4+x)^2}$

$= \frac{-3x^4-24x^3-6}{(x^4+x)^2} < 0$ for $x \geq 1$ $\therefore f$ decreases $\Rightarrow \{b_n\}$ decreases $\Rightarrow \Sigma$ passes A.S.T

(b) $|S - S_n| \leq b_{n+1} < 1/25$

n	$b_n = \frac{n+6}{n^4+n}$
2	$8/18$
3	$9/84 < 9/81 = 1/9$
4	$10/260 = 1/26$ *

$\underline{n=3}$ $|S - S_3| \leq b_4 = \frac{1}{26} < \frac{1}{25}$