EXAM 1 Math 34 (Calc II), Spring 2013

SOLUTIONS

NAME, SECTION

(1)

Sample question: Evaluate $\int \frac{1-x^2}{x(3x^2+10)}\,dx$.

Student's first step: $\frac{1-x^2}{x(3x^2+10)} = \frac{A}{x} + \frac{B}{3x^2+10}$.

(a) What would your first step have been instead?

$$\frac{A}{x} + \frac{Bx+C}{3x^2+10}$$

(b) Explain why the student's first step will lead to trouble in the next few steps. (If you are not sure, carry out the next few steps to find the problem.)

With only two constants (A, B), it might be impossible to solve. Here, the student would find...

$$A(3x^2+10) + Bx = 1-x^2$$
$$\Downarrow$$
$$3Ax^2 + Bx + 10A = -x^2 + 1$$
$$\Downarrow$$
$$3A = -1 \text{ and } 10A = 1.$$

Impossible! (no solutions.)

(2) Write a correct **form** to set up the integrand $\frac{12x^4-100x+1}{(x^3-x)(x^3+x)}$ for partial fractions.

(No need to solve for the constants.)

$$\frac{12x^4-100x+1}{(x^3-x)(x^3+x)} = \frac{12x^4-100x+1}{x^2(x-1)(x+1)(x^2+1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1} + \frac{Ex+F}{x^2+1}$$

(3) Evaluate $\int x\sec^2(x)\,dx$.

$\int u\,dv = uv - \int v\,du$ (int. by parts) Put $u = x$, $du = dx$, $dv = \sec^2 x\,dx$, $v = \tan x$

$$\int x\sec^2 x\,dx = x\tan x \underbrace{- \int \tan x\,dx}$$

$$\int -\tan x\,dx = \int \frac{-\sin x}{\cos x}\,dx = \ln|\cos x| + C$$

$$\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C$$

Final answer: $x\tan x + \ln|\cos x| + C$

(4) Evaluate $\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$.

$\tan^2\theta + 1 = \sec^2\theta$ so put $\tan\theta = \frac{x}{3}$ $(x = 3\tan\theta)$; $dx = 3\sec^2\theta\,d\theta$

Then $\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\tan^2\theta + 1} = 3\sec\theta$.

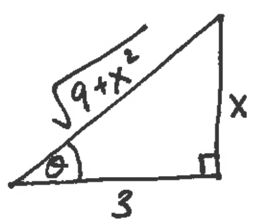
$$\int \frac{1}{x^2\sqrt{x^2+9}}\,dx = \int \frac{1}{\underbrace{(9\tan^2\theta)}_{x^2}\underbrace{(\cancel{3}\sec\theta)}_{\sqrt{x^2+9}}} \cdot \underbrace{\cancel{3}\sec^{\cancel{2}}\theta\,d\theta}_{dx} = \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int u^{-2}\,du = -\frac{1}{9}u^{-1} + C$$

u-Substitution
$u = \sin\theta$
$du = \cos\theta\,d\theta$

$$= \frac{-1}{9\sin\theta} + C$$

$$= \frac{-\sqrt{x^2+9}}{9x} + C$$



$\sin\theta = \frac{x}{\sqrt{x^2+9}}$

Final answer: $\frac{-\sqrt{x^2+9}}{9x} + C$

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(5) **Student work:** $\displaystyle\int_{-2}^{2} \frac{x+2}{x^2+4x+3}\,dx = \left[\frac{1}{2}\ln\left|x^2+4x+3\right|\right]_{-2}^{2} = \frac{1}{2}(\ln 15 - \ln 1) = \frac{\ln 15}{2}$

(a) What's wrong with this solution?

The student hasn't noticed That it's an improper integral. $x^2+4x+3 = (x+3)(x+1)$ has roots at $x=-1,-3$ and $x=-1$ is in the domain of integration.

(b) Fix it: evaluate that integral or show that it diverges.

$$\int_{-2}^{2} \frac{x+2}{x^2+4x+3}\,dx = \underbrace{\int_{-2}^{-1} \frac{x+2}{x^2+4x+3}\,dx}_{\displaystyle \| \atop \displaystyle \lim_{a\to -1^-}\left[\frac{1}{2}\ln\left|x^2+4x+3\right|\right]_{-2}^{a}} + \int_{-1}^{2} \frac{x+2}{x^2+4x+3}\,dx$$

But as $a \to -1^-$, $x^2+4x+3 \to 0^+$, so $\ln|x^2+4x+3| \to -\infty$

So this limit does not exist, which makes the whole integral divergent.

(6) Solve for y from the equation $t+\left(2y\sqrt{t^2+1}\right)\frac{dy}{dt}=0$, given that $y=1$ at time $t=0$.

Separation of variables:

$$2y\sqrt{t^2+1}\,\frac{dy}{dt}=-t \Rightarrow 2y\,dy=\frac{-t}{\sqrt{t^2+1}}\,dt.$$

So $\int 2y\,dy = \int \frac{-t}{\sqrt{t^2+1}}\,dt = \int \frac{-\frac{1}{2}du}{\sqrt{u}} = -u^{1/2}+C_2$

$\boxed{\begin{matrix}u=t^2+1\\ du=2t\end{matrix}}$

$\int 2y\,dy = y^2+C_1$

This gives $y^2=-\sqrt{u}+C = C-\sqrt{t^2+1}$,

so $y=\pm\sqrt{C-\sqrt{t^2+1}}$. When $t=0$,

$y=\pm\sqrt{C-1}=1$

So $C=2$ and the $\pm$ is $+$

Final answer: $y=\sqrt{2-\sqrt{t^2+1}}$

(7) Evaluate $\int \sin^{16}(x)\cos^3(x)\,dx$.

$\sin^2 x+\cos^2 x=1$, so $\cos^2 x = 1-\sin^2 x$.

Thus $\int \sin^{16}(x)\cos^3(x)\,dx = \int \sin^{16}(x)\left(1-\sin^2 x\right)\cos x\,dx$

$$=\int\left(\sin^{16}x-\sin^{18}x\right)\cos x\,dx$$

$\boxed{\begin{matrix}u=\sin x\\ du=\cos x\,dx\end{matrix}}$

$$=\int\left(u^{16}-u^{18}\right)du = \frac{1}{17}u^{17}-\frac{1}{19}u^{19}+C$$

$$=\frac{\sin^{17}(x)}{17}-\frac{\sin^{19}(x)}{19}+C.$$

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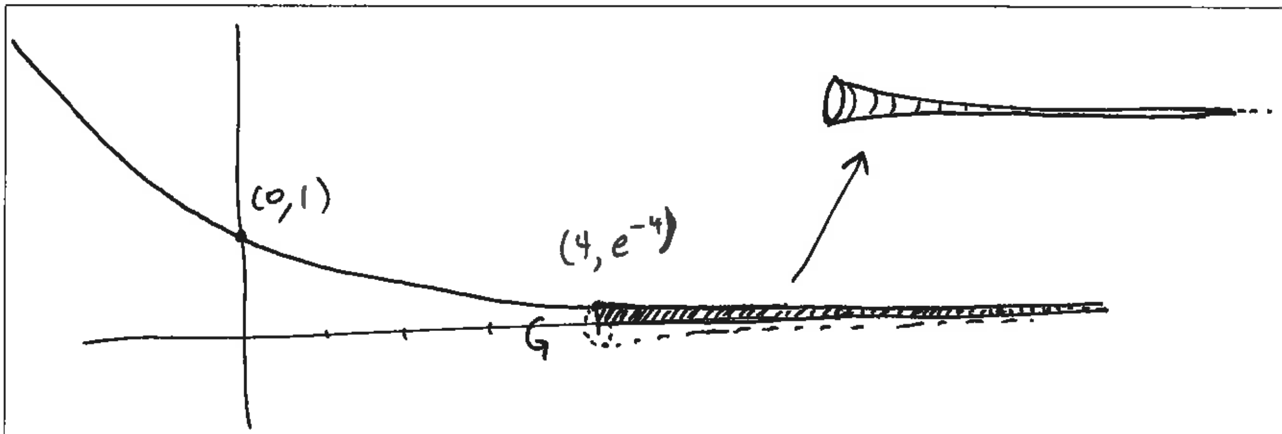
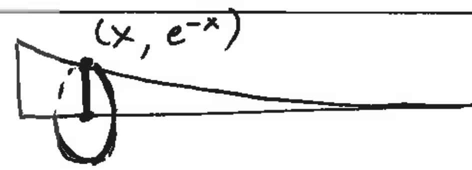
(8) Find the volume of the infinite solid created when e^{-x} is revolved around the x-axis, for $x \geq 4$.

First sketch it.



Then solve it.

Let's use the method of disks.



$r = e^{-x}$, so

$$V = \int_4^\infty \pi r^2\,dx = \int_4^\infty \pi e^{-2x}\,dx = \left[-\frac{\pi}{2}e^{-2x}\right]_4^\infty$$

$$= \lim_{b\to\infty}\left[-\frac{\pi}{2}e^{-2x}\right]_4^b = 0 - \left(-\frac{\pi}{2}e^{-8}\right) = \frac{\pi}{2e^8}$$

Final answer: $\pi/2e^8$

(9) (a) Use integration by parts to evaluate $\int \ln x \, dx$.

$\int u\,dv = uv - \int v\,du$. Put $u = \ln x$, $dv = dx$.

Thus $du = \frac{1}{x}dx$, $v = x$.

So $\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C$.

(b) Now, using your answer from the last part, evaluate $\int (\ln x)^2 \, dx$.

$\int (\ln x)^2 dx = \int (\ln x)(\ln x)\, dx$, <u>NOT</u> $\int 2(\ln x)\, dx$!!

Put $u = \ln x$, $dv = \ln x \, dx$

Then $du = \frac{1}{x}dx$, $v = x\ln x - x$ from above (because this is the antiderivative of $\ln x$)

So $\int (\ln x)^2 dx = \ln x\,(x \ln x - x) - \int (\ln x - 1)\, dx$

$= \ln x\,(x\ln x - x) - \underbrace{(x \ln x - x)}_{\int \ln x\, dx} + \underbrace{x}_{\int 1\, dx} + C$

1	2	3	4	5	6	7	8	9	TOTAL
[10]	[10]	[10]	[10]	[12]	[12]	[10]	[12]	[14]	[100]