KEY

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Calculus II

TUFTS UNIVERSITY
Department of Mathematics

October 5, 2015
Exam I
12-1:20

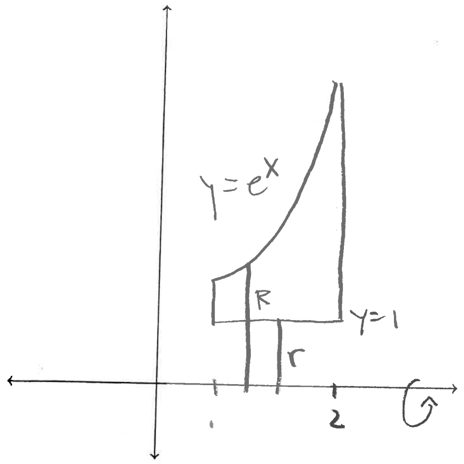
No books, notes, **or calculators**. **TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. You should evaluate standard trigonometric functions like $\tan(\pi/3)$. You **are not allowed to** use reduction formulas. You may not use the tabular method for integration by parts problems. Even when told not to simplify, you are not allowed to leave answers in unsimplified forms like $\cos^{-1}(\sin(2x))$. You are not allowed to quote results about growth rates. You are required to **sign** your exam on the last page. With your signature, you pledge that you have neither given nor received assistance on this exam.

On the inside of the last page there is a blank side of paper for scratch work that is not to be graded.

Problem	Point Value	Points
1	10	
2	18	
3	16	
4	10	
5	10	
6	10	
7	10	
8	16	
	100	

1. (10 points) Let R be the region bounded by the curve $y = e^x$, the horizontal line y=1, and the vertical lines $x = 1$ and $x = 2$.

(a) Sketch the region R. Set up but *do not evaluate* an integral that gives the volume of the solid obtained by rotating the region R about the x-axis.

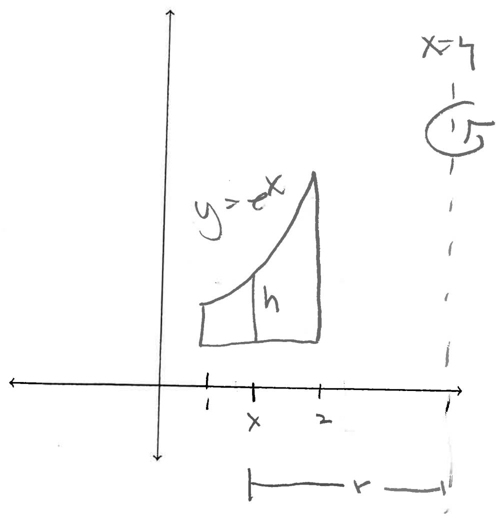


Washers

$$V = \pi \int_1^2 R^2 - r^2 \, dx$$

$$= \pi \int_1^2 \left[(e^x)^2 - 1^2\right] dx$$

(b) Sketch the same region R and the vertical line $x = 4$. Set up but *do not evaluate* an integral that gives the volume of the solid obtained by rotating the region R about the vertical line $x = 4$.

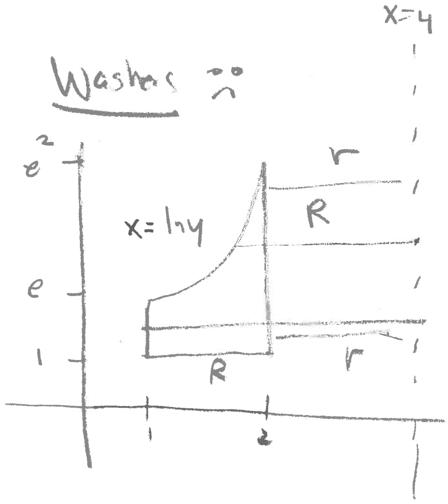


Shells

$$V = 2\pi \int_1^2 r h \, dx$$

$$= 2\pi \int_1^2 (4-x)(e^x - 1) \, dx$$

Washers :(



$$V = \pi \int_1^e \left((4-1)^2 - (4-2)^2\right) dy$$

$$+ \pi \int_e^{e^2} \left((4 - \ln y)^2 - (4-2)^2\right) dy$$

2. (18 points) Short answers. (You may use known formulas.)

(a) $\int \tan x \, dx$ $= \ln|\sec x| + c$

or

$= -\ln|\cos x| + c$

(b) $\int \sec x \, dx$ $= \ln|\sec x + \tan x| + c$

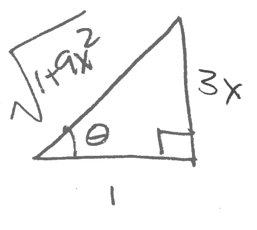
(c) $\int \frac{dx}{x^2+1}$ $= \tan^{-1} x + C$

(d) Evaluate and simplify your answer: $\int_2^4 \frac{dx}{x-5}$ $= \ln|x-5| \Big|_2^4$

$= \ln|4-5| - \ln|2-5|$

$= \underbrace{\ln 1}_{0} - \ln 3 = \boxed{-\ln 3}$

(e) The value of $\cos\theta$ if $\tan\theta = 3x$ is $\frac{1}{\sqrt{1+9x^2}}$



(f) The u substitution required to evaluate $\int \frac{\cos(\ln x)}{x} dx$ is

$u = \ln x$

$\left(du = \frac{dx}{x}\right)$

3. (16 points) Evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/8} \cos^2 x\, dx = \frac{1}{2}\int_0^{\pi/8}(1+\cos 2x)\,dx = \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]\Big|_0^{\pi/8}$

$= \frac{1}{2}\left[\left(\pi/8 + \frac{1}{2}\sin\frac{\pi}{4}\right) - (0+0)\right]$

$= \frac{1}{2}\left(\pi/8 + \frac{1}{2}(\sqrt{2}/2)\right) = \frac{\pi}{16} + \frac{\sqrt{2}}{8}$

(b) $\displaystyle\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx$

$= \int (1-\cos^2 x)\sin x\, dx = -\int (1-u^2)\,du$

$\left(\begin{array}{c} u = \cos x \\ du = -\sin x\, dx \\ -du = \sin x\, dx \end{array}\right)$

$= -\left[u - u^3/3\right] + c$

$= -\left[\cos x - \frac{\cos^3 x}{3}\right] + c$

$= \frac{\cos^3 x}{3} - \cos x + C$

(c) $\displaystyle\int \tan^2 x\, dx = \int(\sec^2 x - 1)\,dx = \tan x - x + C$

4. (10 points) Evaluate. (Note that $(\ln x)^2 \neq 2\ln x$.)

$$\int x^2(\ln x)^2\, dx$$

PARTS: $u = (\ln x)^2$ $\quad dv = x^2\,dx$

$du = \frac{2\ln x}{x}\,dx$ $\quad v = \frac{x^3}{3}$

$$= \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\int (\ln x)\, x^2\, dx$$

PARTS: $u = \ln x$ $\quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{x^3}{3}$

$$= \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\left[\frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx\right]$$

$$= \frac{x^3}{3}(\ln x)^2 - \frac{2}{3}\left[\frac{x^3}{3}\ln x - \frac{1}{3}\left(\frac{x^3}{3}\right)\right] + C$$

$$= \frac{x^3}{3}(\ln x)^2 - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C$$

5. (10 points) Evaluate.

$$\int \frac{dt}{(4-t^2)^{3/2}}$$

<u>TRIG SUB</u> $t = 2\sin\theta$

$dt = 2\cos\theta \, d\theta$

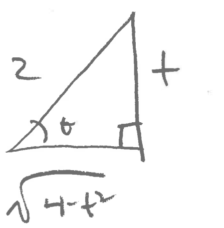
$$(4-t^2)^{3/2} = (4-4\sin^2\theta)^{3/2} = 4^{3/2}(1-\sin^2\theta)^{3/2} = 4^{3/2}(\cos^2\theta)^{3/2}$$

$$= 8\cos^3\theta$$

$$= \int \frac{2\cos\theta \, d\theta}{8\cos^3\theta} = \frac{1}{4}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\int \sec^2\theta \, d\theta$$

$$= \frac{1}{4}\tan\theta + C = \boxed{\frac{1}{4}\frac{t}{\sqrt{4-t^2}} + C}$$

$\sin\theta = t/2$



6. (10 points) Evaluate

$$\int \frac{8(x^2+4)}{x(x^2+8)}\,dx$$

PF's: $\dfrac{8(x^2+4)}{x(x^2+8)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+8}$

$$8(x^2+4) = A(x^2+8) + (Bx+C)x$$

$$8x^2+32 = Ax^2 + 8A + Bx^2 + Cx$$

$32 = 8A \Rightarrow A=4$

$0 \cdot x = Cx \Rightarrow C=0$

$8x^2 = Ax^2 + Bx^2$

$8 = A+B$

$8 = 4+B \Rightarrow B=4$

$$\int \frac{4\,dx}{x} + \int \frac{4x\,dx}{x^2+8} = \int \frac{4\,dx}{x} + 2\int \frac{du}{u}$$

$u = x^2+8$

$du = 2x\,dx$

$$= 4\ln|x| + 2\ln|u| + C$$

$$= \boxed{4\ln|x| + 2\ln|x^2+8| + C}$$

7. (10 points) Evaluate

$$\int \frac{x^3\,dx}{\sqrt{1+9x^2}}$$

TRIG SUB

$x = \frac{1}{3}\tan\theta$

$dx = \frac{1}{3}\sec^2\theta\,d\theta$

$\sqrt{1+9x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$

$$= \int \frac{\frac{1}{27}\tan^3\theta \;\; \frac{1}{3}\sec^2\theta\,d\theta}{\sec\theta} = \frac{1}{81}\int \tan^3\theta\sec\theta\,d\theta$$

$$= \frac{1}{81}\int \tan^2\theta\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{81}\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$

$u = \sec\theta$

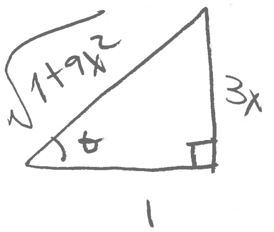
$du = \sec\theta\tan\theta\,d\theta$

$$= \frac{1}{81}\int (u^2-1)\,du = \frac{1}{81}\left[u^3/3 - u\right] + C$$

$$= \frac{1}{81}\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$\tan\theta = 3x$

$$= \frac{1}{81}\left[\frac{1}{3}\left(\sqrt{1+9x^2}\right)^3 - \left(\sqrt{1+9x^2}\right)\right] + C$$



8. (16 points) Evaluate each of the following improper integrals or show that it diverges.

(a) $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ $= \lim_{a \to 0^+} \int_a^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \lim_{a \to 0^+} e^{\sqrt{x}} \Big|_a^4$

$(0, 4]$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int e^u du$$

$u = \sqrt{x}$ $\qquad = 2e^u + c$

$du = \frac{1}{2\sqrt{x}} dx$ $\qquad = 2e^{\sqrt{x}} + c$

$2du = \frac{dx}{\sqrt{x}}$

$= 2 \lim_{a \to 0^+} [e^{\sqrt{4}} - e^{\sqrt{a}}]$ ($\sqrt{a} \to 0$)

$= 2(e^2 - e^0) = 2(e^2 - 1)$

Integral converges to $2(e^2-1)$

(b) $\int_0^\infty \frac{x}{\sqrt[5]{x^2+1}} dx$ $= \lim_{b \to \infty} \int_0^b \frac{x\,dx}{\sqrt[5]{x^2+1}} = \frac{5}{8} \lim_{b \to \infty} \left[(x^2+1)^{4/5}\right] \Big|_0^b$

$$\int \frac{x\,dx}{(x^2+1)^{1/5}} = \frac{1}{2}\int u^{-1/5} du$$

$u = x^2+1$ $\qquad = \frac{1}{2}\left(\frac{5}{4} u^{4/5}\right) + c$

$du = 2x\,dx$ $\qquad = \frac{5}{8} u^{4/5} + c$

$\frac{1}{2} du = x\,dx$ $\qquad = \frac{5}{8}(x^2+1)^{4/5} + c$

$= \frac{5}{8} \lim_{b \to \infty} \left[(b^2+1)^{4/5} - 1\right]$

$\downarrow$

∞

This integral <u>diverges</u>