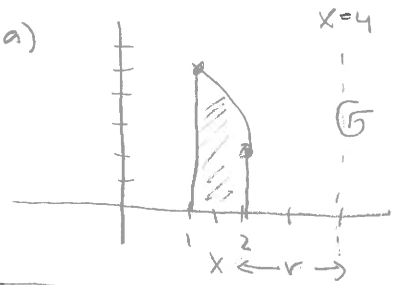
MATH 34 Exam I Solutions F '13

1 (8) a)



b) $V = 2\pi \int_1^2 r\,h\,dx$

$= 2\pi \int_1^2 (4-x)(6-x^2)\,dx$

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2 (10) $\displaystyle\int \frac{\sin^3 x\,dx}{\sqrt{\cos x}} = \int \frac{\sin^2 x \sin x\,dx}{\sqrt{\cos x}} = \int \frac{(1-\cos^2 x)\sin x\,dx}{\sqrt{\cos x}}$

Let $u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$\displaystyle = -\int \frac{(1-u^2)}{\sqrt{u}}\,du = -\int\left(\frac{1}{u^{1/2}} - \frac{u^2}{u^{1/2}}\right)du = -\int\left(u^{-1/2} - u^{3/2}\right)du$

$\displaystyle = -\left[2\sqrt{u} - \frac{2}{5}u^{5/2}\right] + C = -\left[2\sqrt{\cos x} - \frac{2}{5}(\cos x)^{5/2}\right] + C$

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3 (10) $\displaystyle\int_2^4 x\ln x\,dx = \frac{1}{2}x^2\ln x\Big|_2^4 - \frac{1}{2}\int_2^4 \frac{x^2}{x}\,dx = \frac{1}{2}x^2\ln x\Big|_2^4 - \frac{1}{2}\int_2^4 x\,dx$

(parts $u = \ln x$ $dv = x\,dx$
$du = \frac{1}{x}dx$ $v = \frac{x^2}{2}$)

$\displaystyle = \frac{1}{2}x^2\ln x\Big|_2^4 - \frac{1}{2}\left(\frac{x^2}{2}\right)\Big|_2^4$

$\displaystyle = \frac{1}{2}\left[(16)\ln 4 - 4\ln 2\right] - \frac{1}{4}\left[16-4\right]$

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4 (10) $\displaystyle\int\sqrt{4-x^2}\,dx = 4\int\cos^2\theta\,d\theta = \frac{4}{2}\int(1+\cos 2\theta)\,d\theta$

($\boxed{x = 2\sin\theta}$, $-\pi/2 \le \theta \le \pi/2$
$dx = 2\cos\theta\,d\theta$
$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta}$
$= |2\cos\theta| = 2\cos\theta$)

$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

$= 2\left[\theta + \frac{1}{2}(2\sin\theta\cos\theta)\right] + C$

$= 2\left[\theta + \sin\theta\cos\theta\right] + C$

$\displaystyle = 2\left[\sin^{-1}\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right] + C$

$\sin\theta = \frac{x}{2}$

$\boxed{5}$ (10) $\int \tan^3(2x)\sec(2x)\,dx = \int \tan^2(2x)\sec(2x)\tan(2x)\,dx$

$= \int (\sec^2(2x)-1)\sec(2x)\tan(2x)\,dx$ $\left(\begin{array}{l}\text{Let } u = \sec 2x \\ du = 2\sec 2x \tan 2x\,dx \\ \frac{1}{2}du = \sec 2x \tan 2x\,dx\end{array}\right)$

$= \frac{1}{2}\int (u^2-1)\,du = \frac{1}{2}\left[\frac{u^3}{3} - u\right] + C$

$= \frac{1}{2}\left[\frac{\sec^3 2x}{3} - \sec 2x\right] + C$

$\boxed{6}$ (10) $\int \frac{7x^2+x+3}{x^3+x^2}\,dx = \int \frac{7x^2+x+3}{x^2(x+1)}\,dx = \int \frac{-2\,dx}{x} + \int \frac{3\,dx}{x^2} + \int \frac{9\,dx}{x+1}$

PF's $\frac{7x^2+x+3}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

(*) $7x^2+x+3 = Ax(x+1) + B(x+1) + Cx^2$

$x=0$ $3 = B$

$x=-1$ $9 = C$

x^2: $7 = A + C \to A = -2$

$= -2\ln|x| - \frac{3}{x} + 9\ln|x+1| + C$

$\boxed{7}$ (10) a) $\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^t$

$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du$

$u = -x^2$ $= -\frac{1}{2}e^u + C$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$ $= -\frac{1}{2}e^{-x^2} + C$

$= -\frac{1}{2}\lim_{t\to\infty}\left(\frac{1}{e^{x^2}}\right)\Big|_0^t$

$= -\frac{1}{2}\lim_{t\to\infty}\left[\underbrace{\frac{1}{e^{t^2}}}_{0} - \frac{1}{e^0}\right] = -\frac{1}{2}(1)$

$= \frac{1}{2}$

Integral converges to $\frac{1}{2}$

$\boxed{7}$ (6) $\displaystyle\int_0^4 \frac{dx}{\sqrt[3]{8-2x}} = \lim_{t\to 4^-} \int_0^t \frac{dx}{\sqrt[3]{8-2x}}$

$[0,4)$

$\displaystyle\int \frac{dx}{\sqrt[3]{8-2x}} = -\frac{1}{2}\int u^{-1/3}\,du$

$u = 8-2x$ $\quad = -\frac{1}{2}\left(\frac{3}{2}\right)u^{2/3} + C$

$du = -2\,dx$ $\quad = -\frac{3}{4}(8-2x)^{2/3} + C$

$-\frac{1}{2}du = dx$

$\displaystyle = \lim_{t\to 4^-} -\frac{3}{4}\left[(8-2x)^{2/3}\right]\Big|_0^4$

$\displaystyle = -\frac{3}{4}\lim_{t\to 4^-}\left[(8-2t)^{2/3} - (8)^{2/3}\right]$

$(8-2t)^{2/3} \to 0$

$= \left(\frac{-3}{4}\right)(-4) = 3$

Integral converges to 3

$\boxed{8}$ (10) $\dfrac{dy}{dt} = 2y-6$, $y(0)=8$

$\displaystyle\int \frac{dy}{2y-6} = \int dt$

$u = 2y-6$

$du = 2dy$

$\frac{1}{2}du = dy$

$\displaystyle\frac{1}{2}\int\frac{du}{u} = \int dt$

$\frac{1}{2}\ln|u| = t + C$

$\frac{1}{2}\ln|2y-6| = t+C$

rename C

$\ln|2y-6| = 2t+C$

$|2y-6| = e^{2t+C} = e^C e^{2t}$

rename C

$2y-6 = Ce^{2t}$ $\quad (C\neq 0)$

$2y = Ce^{2t}+6$

$y = Ce^{2t}+3$ (rename C)

Solve for C:

$8 = Ce^{2\cdot 0}+3$

$8 = C+3$

$C=5$

$\boxed{y = 5e^{2t}+3}$

$\boxed{9}$ a) (2) $a_{n+1} = 2a_n$, $n\geq 0$, $a_0 = 3$

(i) $3,\ 2\cdot3,\ 2\cdot2\cdot3,\ 2\cdot2\cdot2\cdot3$

$3,\ 6,\ 12,\ 24$

(ii) $a_n = 3\cdot 2^n$, $n\geq 0$

(b) (6) $\displaystyle\lim_{n\to\infty} n e^{-2n} = \lim_{n\to\infty}\frac{n}{e^{2n}} = \lim_{x\to\infty}\frac{x}{e^{2x}}$

$n\to\infty$, $e^{-2n}\to 0$; $\frac{\infty}{\infty}$

$\displaystyle\overset{L}{=} \lim_{t\to\infty}\frac{1}{2e^{2x}} = 0$ Seq. converges to 0

(c) (6) $a_n = \left(\frac{3}{n}\right)^{\frac{1}{n}}$

$\boxed{1}$ $\displaystyle\lim_{n\to\infty}\ln\left(\frac{3}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{n}\ln\frac{3}{n} = \lim_{n\to\infty}\frac{\ln\frac{3}{n}}{n}$

$\frac{1}{n}\to 0$, $\ln\frac{3}{n}\to -\infty$; $\frac{\to -\infty}{\to\infty}$

$\displaystyle = \lim_{x\to\infty}\frac{\ln 3/x}{x} \overset{L}{=} \lim_{x\to\infty}\frac{1}{\left(\frac{3}{x}\right)}\left(\frac{-1}{x^2}\right)$

$\displaystyle = \lim_{x\to\infty}\left(\frac{x}{3}\right)\left(\frac{-1}{x^2}\right) = \lim_{x\to\infty}\frac{-1}{3x} = 0$

$\boxed{2}$ Seq. converges to $e^0 = 1$.