Tufts University

Math 12 | Department of Mathematics | December 14, 2007

Final Exam | 8:30 – 10:30

No books, notes, cell phones, **or calculators**. The exam has three pages, please make sure you have all of them. In all problems you must show all your work in the bluebook, giving **full justification**. Cross out any work you do not want graded. You are required to **sign** your exam book; with your signature, you pledge that you have neither given nor received assistance on this exam. Final grades will be available on SIS 24 hours after they have been posted by your instructor.

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1. (9 points)

(a) Find the two complex numbers z which are solutions to the equation

$$2z^2 - 4z + 3 = 0.$$

(b) Express the complex number $z = e^{1+5\pi i/3}$ in the form $z = a + bi$ for real numbers a, b.

c) Express $z = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{60}$ in the form $z = a + bi$ for real numbers a, b.

2. (24 points) Evaluate the following integrals:

(a) $\displaystyle\int \frac{x^2 + 8x + 7}{(x-1)(x^2+6x+9)}\,dx$ **(b)** $\displaystyle\int \frac{1}{(1+4x^2)^{3/2}}\,dx$ **(c)** $\displaystyle\int x \ln x\,dx$

3. (8 points) Using the method of cylindrical shells, set up, but do not evaluate, an integral or integrals for the volume generated by revolving the region bounded by the curves $y = \ln x$, $y = 0$, $x = e$ about the y-axis. Be sure to draw a picture of the region you are revolving.

4. (8 points) Without using series solutions, solve the initial value problem:

$$\frac{dy}{dx} = \frac{2x + \sec^2 x}{2y}, \qquad y(0) = -5.$$

Express your answer as $y = f(x)$, i.e. solve for y explicitly.

EXAM CONTINUES ON REVERSE.

5. (12 points)

(a) (5 pts) Determine whether the series $\sum_{n=1}^{\infty} \frac{(\sin^2 n)\sqrt{n}}{n^3+1}$ converges or diverges.

(b) (7 pts) Find the *radius of convergence* R and the *interval of convergence* I of the power series

$$\sum_{n=2}^{\infty} \frac{2^n x^n}{\ln n}.$$

6. (5 points) A sequence of numbers is given by the following recursion relation:

$$a_{n+1} = \frac{a_{n-1}}{n} \quad \text{for } n \geq 1, \quad a_0 = 0, \ a_1 = 3.$$

Solve for a_n.

7. (9 points) You may use the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x in answering the following questions.

(a) (3 pts) Choose one: The absolute value of the error made in approximating e^{-2} by the 4th degree Taylor polynomial about 0 evaluated at $x = -2$, $T_4(-2)$, is $\begin{cases} \text{larger than} \\ \text{less than} \\ \text{equal to} \end{cases}$ 1/3.

In justifying your answer, you may assume the series for e^{-2} satisfies the conditions of the Alternating Series Theorem.

(b) (6 pts)

(i) Use a change of indices to help you evaluate the following infinite series: $\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{(n-2)!}$.

(ii) Determine the function equal to the following power series: $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n\, n!}$.

8. (10 points)
Consider the curve given by the parameric equations

$$C = \left\{ (x,y) : x = \sin t, \ y = e^{\sin t} \right\}.$$

(a) (3 pts) Eliminate the parameter t to find a Cartesian equation of the curve.

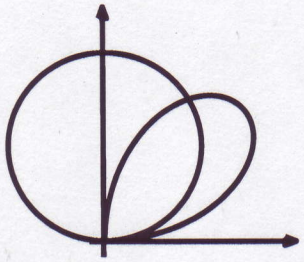
(b) (7 pts) Sketch the curve C and indicate with arrows the path in which it is traversed as t increases from $t = -\pi/2$ to $t = \pi$. Be sure to label the starting and ending points.

EXAM CONTINUES ON NEXT PAGE.

9. (7 points) **(a)** (3 pts) Below are graphs of the two polar curves

$$C_1 : r = \sin\theta,\ 0 \le \theta \le \pi \qquad \text{and} \qquad C_2 : r = \sin(2\theta),\ 0 \le \theta \le \pi/2.$$

Copy the graphs in your blue book. Label the curves and indicate the polar coordinates of the points at which they intersect.

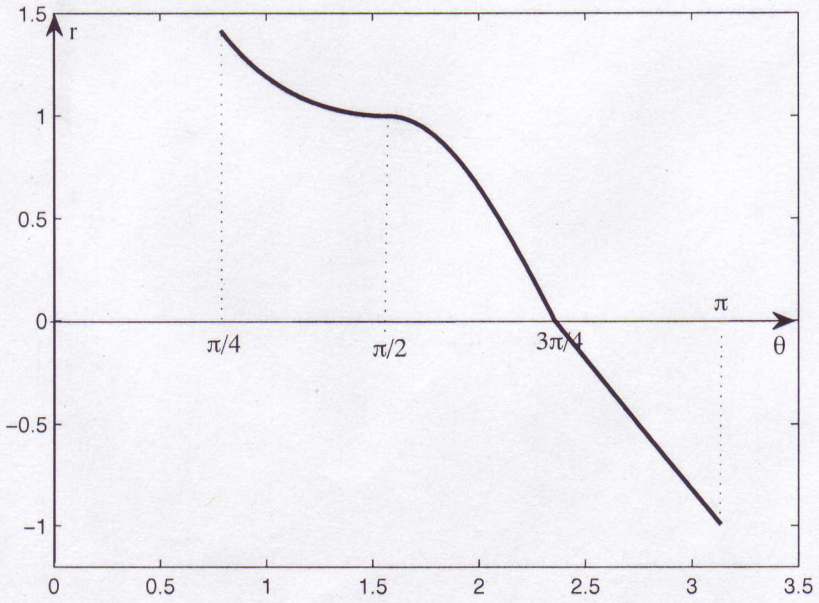


(b) (4 pts) Set-up, but do not evaluate, an integral or integrals for the area which is inside C_2 but outside C_1.

10. (8 points) Do this problem on a new page.

(a) (2 pt) Identify the polar curve: $r\sin\theta = 1$ by changing the equation to (x, y) Cartesian coordinates.

(b) (6 pts) Below is the (θ, r) Cartesian graph of a polar curve. The portion for $\pi/4 \le \theta \le \pi/2$ corresponds to a portion of the curve you identified in part (a). (The remainder of the graph corresponds to a different polar curve.)



Using this graph together with what you know from part (a), graph the polar curve in (r, θ) polar coordinates. Make the drawing large with 4 bluebook squares per 1 unit. Label all points at which your polar curve intersects the Cartesian axes (i.e. the $x - y$ axes); use (r, θ)-coordinates for your labels.

END OF EXAM.