Math 11 Final Exam Spring 2011 Solution

1 (a) $u = \sin(2x)$
$du = 2\cos(2x)\,dx$
$\frac{1}{2}du = \cos(2x)\,dx$

$\Rightarrow \int \frac{\cos(2x)}{\sin(2x)}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|\sin(2x)| + C$

(b) $u = x+1$ $\quad x = u-1$
$du = dx$

$\int \frac{x^2}{\sqrt{x+1}}\,dx = \int \frac{u^2-2u+1}{\sqrt{u}}\,du = \int \left(u^{3/2} - 2u^{1/2} + u^{-1/2}\right)du$

$= \frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2} + 2u^{1/2} + C$

$= \frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + 2(x+1)^{1/2} + C$

2 $a(t) = 2 + e^{t-1}$ meters/second^2

$v(t) = \int a(t)\,dt = 2t + e^{t-1} + C$

$2 = v(1) = 2 + 1 + C \Rightarrow C = -1$

$v(t) = 2t + e^{t-1} - 1$ meters/second

$s(t) = \int v(t)\,dt = t^2 + e^{t-1} - t + D$

$.2 = s(1) = 1 + 1 - 1 + D \Rightarrow D = .2 - 1 = -.8$

$\boxed{s(t) = t^2 + e^{t-1} - t - .8}$ meters

3 (a) $= \int_e^{e^3}\left(\frac{1}{2x} + 1\right)dx = \frac{1}{2}\int_e^{e^3}\frac{1}{x}\,dx + \int_e^{e^3} dx$

$= \frac{1}{2}\ln|x|\Big|_e^{e^3} + x\Big|_e^{e^3} = \frac{1}{2}(3-1) + (e^3 - e)$

$= \boxed{1 + e^3 - e}$

$\boxed{3}$ (b) $\displaystyle\int_{-1}^{1} t^2 \cos(\pi t^3)\,dt$

$u = \pi t^3$

$du = 3\pi t^2\,dt$

$\frac{1}{3\pi}\,du = t^2\,dt$

$$= \frac{1}{3\pi}\int_{u(-1)}^{u(1)} \cos(u)\,du = \frac{1}{3\pi}\int_{-\pi}^{\pi} \cos(u)\,du$$

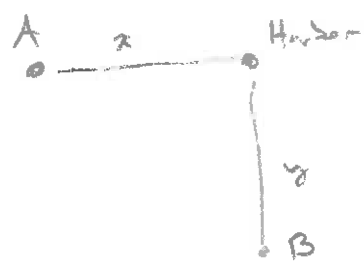
$$= \frac{1}{3\pi}\sin(u)\Big|_{-\pi}^{\pi} = \frac{1}{3\pi}(0-0) = \boxed{0.}$$

since $\frac{1}{x^2} = \frac{1}{\sqrt{x^4}}$ for $x >> 0$

$\boxed{4}$ (a) $$\lim_{x\to\infty} \frac{x+1}{\sqrt{x^4+x+1}} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\overset{\to 0}{1/x} + \overset{\to 0}{1/x^2}}{\sqrt{1 + \underset{\to 0}{1/x^3} + \underset{\to 0}{1/x^4}}} = \frac{0}{1} = \boxed{0}$$

(b) $$\lim_{x\to\infty} \frac{\overset{\to\infty}{\ln(x)}}{\underset{\to\infty}{\ln(2x+1)}} \overset{\text{l'Hopital}}{=\!=} \lim_{x\to\infty} \frac{1/x}{\left(\frac{1}{2x+1}\right)\cdot 2} = \frac{1}{2}\lim_{x\to\infty} \frac{2x+1}{x}$$

$$= \frac{1}{2}\lim_{x\to\infty}\left(2 + \frac{1}{x}\right) = \frac{1}{2}\cdot 2 = \boxed{1}$$



$\boxed{5}$ Let x = distance from A to Harbor, and
y = distance from B to Harbor
t = hours after noon

$\frac{dx}{dt} = -5$ mi/hr, $\frac{dy}{dt} = 4$ mi/hr

$x(1) = 15 - 5 = 10$ mi $y(1) = 6 + 4 = 10$ mi

ℓ = distance from A to B. $\Rightarrow \ell^2 = x^2 + y^2$. $\ell(1)^2 = 2\cdot 10^2 = 200$ mi

$\ell(1) = 10\sqrt{2}$.

$$\Rightarrow 2\ell\frac{d\ell}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

when $t=1$ $\quad 2\cdot 10\sqrt{2}\,\frac{d\ell}{dt} = 20\cdot(-5) + 20\cdot(4) \Rightarrow \boxed{\frac{d\ell}{dt}} = \frac{-20}{2\cdot 10\sqrt{2}} = \boxed{\frac{-1}{\sqrt{2}}\text{ mi/hr}}$

$\boxed{6}$ (a) $$\frac{dy}{dx} = \boxed{\frac{\frac{x+2}{x+1} - \ln(x+1)}{(x+2)^2}}$$

(b) $$\frac{dy}{dx} = \boxed{\frac{1}{2}\left(e^{\sin(x)}+1\right)^{-1/2} \cdot e^{\sin(x)} \cdot \cos(x)}$$

(c) $$\frac{d}{dx}\int_0^x t^2\,dt = x^2$$

$$\Rightarrow \frac{d}{dx}\int_0^{\ln(x)} t^2\,dt = [\ln(x)]^2 \cdot \frac{d}{dx}\ln(x) = \boxed{[\ln(x)]^2 \cdot \frac{1}{x}}$$

(d) $$\ln y = x \cdot \ln(3x+1)$$

$$\Rightarrow \frac{y'}{y} = \ln(3x+1) + \frac{x}{3x+1} \cdot 3$$

$$\Rightarrow \boxed{y' = (3x+1)^x\left[\ln(3x+1) + \frac{3x}{3x+1}\right]}$$

$\boxed{7}$ (a) $$xy' + y = 2x + 2yy'$$

$$\Rightarrow y - 2x = (2y - x)y'$$

$$\Rightarrow y' = \frac{y-2x}{2y-x}$$

(b) $$y'\Big|_{(x,y)=(1,1)} = \frac{1-2}{2-1} = \frac{-1}{1} = -1$$

$$y - 1 = (-1)(x-1) \Rightarrow y - 1 = -x + 1$$

$$\boxed{y = -x + 2}$$

$\boxed{8}$ (a) $-x^2 + 2 = 2x - 1$

$\Rightarrow 0 = x^2 + 2x - 3 \quad = (x+3)(x-1)$

The curves intersect when $x = 1$ and $x = -3$.

So $\underline{\underline{\text{area}}} = \int_{-3}^{1} \left[(-x^2+2) - (2x-1)\right] dx$.

(b) $\int_1^4 \sqrt{9-(x-1)^2}\, dx \qquad u = x-1, \quad du = dx$

$= \int_{u(1)=0}^{u(4)=3} \sqrt{9-u^2}\, du$

$= \frac{1}{4}\left(\begin{array}{c}\text{Area of circle}\\ \text{of radius } 3\end{array}\right) = \boxed{\frac{1}{4}\pi \cdot 9}$

$\boxed{9}$ (a) $f'(x) = 3x^2 - 3 \quad = 3(x^2-1) = 3(x-1)(x+1)$

So critical #s of f are $x = 1, -1$.

$x=0 \quad f(0) = 3$

$x=1 \quad f(1) = 1 \qquad \Rightarrow$ abs. minimum of f on $[0,2]$ is $1 = f(1)$.

$x=2 \quad f(2) = 8-6+3 = 5$

(b) $h'(x) = e^x - 2x \qquad h''(x) = e^x - 2$

$h''(x) > 0 \iff e^x > 2 \iff x > \ln(2)$.

So: concave up on $(\ln(2), \infty)$

$\boxed{10}$ $V = 54\pi = \pi r^2 h \Rightarrow h = \frac{54\pi}{\pi r^2} = \frac{54}{r^2}$

Surface Area $S = 2 \cdot \pi r^2 + (2\pi r) \cdot h$

$$S = 2\pi r^2 + 2\pi r \cdot \frac{54}{r^2} = 2\pi r^2 + \frac{4 \cdot \pi \cdot 27}{r}$$

$$\frac{dS}{dr} = 4\pi\left[r - \frac{27}{r^2}\right] \stackrel{?}{=} 0 \iff r^3 = 27 \iff \boxed{r=3}$$

$$\left.\frac{d^2S}{dr^2}\right|_{r=3} = \left. 4\pi \cdot \left\{1 + 2 \cdot \frac{27}{r^3}\right]\right|_{r=3} > 0$$

$\Rightarrow r=3$ minimizes S.

So dimensions giving minimal volume are

$\boxed{r=3 \text{ cm}}$, $\boxed{h =} \frac{54}{9} = \boxed{6 \text{ cm}}$