Math 11 | **Tufts University** | **Spring 2009**

Department of Mathematics

Exam 2 - Solutions

1. (12 pts) Find the derivative. Do not simplify your answer.

   (a) $y = x\tan^{-1} x$ (b) $y = \dfrac{1}{2}\ln(1+x^2)$ (c) $y = x^{\sin x}$

   SOLUTION:

   (a) Using Product Rule: $y' = \dfrac{x}{1+x^2} + \tan^{-1} x$.

   (b) Using Chain Rule: $y' = \dfrac{1}{2}\dfrac{2x}{1+x^2} = \dfrac{x}{1+x^2}$.

   (c) Using Logarithmic Differentiation: $\ln y = \ln x^{\sin x} = \sin x \ln x$.
   Differentiate both sides: $\dfrac{y'}{y} = \cos x \ln x + \dfrac{\sin x}{x}$.
   Solving for y': $y' = y(\cos x \ln x + \dfrac{\sin x}{x}) = x^{\sin x}(\cos x \ln x + \dfrac{\sin x}{x})$.

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2. (6 pts) Evaluate the following.

   (a) $\tan^{-1}(1)$. (b) $\cos^{-1}(\cos\left(\frac{13\pi}{6}\right))$.

   SOLUTION:

   (a) $\tan^{-1}(1) = \dfrac{\pi}{4}$.

   (b) $\cos^{-1}(\cos\left(\frac{13\pi}{6}\right)) = \frac{\pi}{6}$. (The corresponding angle in the range $[0, \pi]$.)

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3. (12 pts) Find the limit.

   (a) $\displaystyle\lim_{x\to 1}\frac{\sin(2x-2)}{\sin(3x-3)}$. (b) $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1}{\sin^{-1} x}$. (c) $\displaystyle\lim_{x\to\infty}\left(1-\frac{7}{x}\right)^x$.

   SOLUTION:

   (a) $\displaystyle\lim_{x\to 1}\frac{\sin(2x-2)}{\sin(3x-3)} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 1}\frac{2\cos(2x-2)}{3\cos(3x-3)} = \frac{2}{3}$.

   (b) $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1}{\sin^{-1} x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0}\frac{2e^{2x}}{\frac{1}{\sqrt{1-x^2}}} = 2$.

   (c) $\displaystyle\lim_{x\to\infty}\left(1-\frac{7}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1-\frac{7}{x}\right)^x} = e^{\lim_{x\to\infty} x\ln\left(1-\frac{7}{x}\right)}$.

   Now we have:

   $$\lim_{x\to\infty} x\ln\left(1-\frac{7}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{7}{x}\right)}{1/x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{\frac{1}{1-\frac{7}{x}}\cdot 7/x^2}{-1/x^2} = -7\lim_{x\to\infty}\frac{1}{1-\frac{7}{x}} = -7.$$

   Therefore the limit of the original is e^{-7}.

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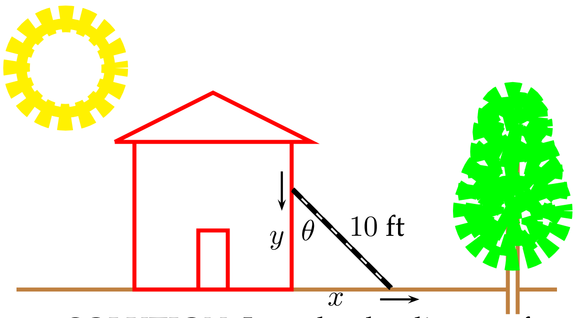
4. (12 pts) A ladder 10 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

   (a) How fast is the top of the ladder moving down the wall when its base is 6 feet from the wall?

   (b) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 6 feet from the wall.



SOLUTION: Let x be the distance from the wall to the bottom of the ladder and y the distance from the top of the ladder to the ground as shown. We are given that $\dfrac{dx}{dt} = 2$.

   (a) The quantities x and y are related by the Pythagorean Theorem: $x^2+y^2 = 100$. Differentiating both sides with respect to time t gives $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$. Since, $\dfrac{dx}{dt} = 2$ we solve and get $\dfrac{dy}{dt} = -2\dfrac{x}{y}$. By the Pythagorean Theorem, when $x = 6$, $y = 8$. So, substituting we get $\dfrac{dy}{dt} = -\dfrac{3}{2}$. Therefor the ladder is moving down at a rate of $3/2$ feet per second.

   (b) The quantities x and θ are related by the trigonometric definition: $\sin\theta = \dfrac{x}{10}$. Differentiating both sides with respect to time t gives $\cos\theta\dfrac{d\theta}{dt} = \dfrac{1}{10}\dfrac{dx}{dt}$. Since, $\dfrac{dx}{dt} = 2$ we solve to get $\dfrac{d\theta}{dt} = \dfrac{1}{5}\sec\theta$. Using the triangle we find that at the given instant, $\sec\theta = \dfrac{10}{8}$, thus $\dfrac{d\theta}{dt} = \dfrac{1}{4}$ radians per second.

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5. (8 pts) For $xy - 2y^2 = -2$, find the equation of the tangent line to the curve at the point $(3, 2)$.
SOLUTION:
Using implicit differentiation (with respect to x): $y + xy' - 4yy' = 0$. Solve for y' to get $y' = \dfrac{-y}{x - 4y}$. This gives the slope of the tangent line at $(3, 2)$ to be $\frac{-2}{-5} = \frac{2}{5}$. Thus an equation of the line is $y - 2 = \frac{2}{5}(x - 3)$ or $y = \frac{2}{5}x + \frac{4}{5}$.

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6. (12 pts) For each function below: (i) Find all critical numbers. (ii) Find the function's absolute maximum and minimum on the given closed interval.

   (a) $f(x) = x^3 - 3x - 1$, $[0, 3]$. (b) $f(x) = 3x^{\frac{2}{3}} - x$, $[-1, 27]$.

   SOLUTION:

   (a) (i) $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$.
   Therefore f has critical numbers $x = -1$ and $x = 1$.
   (ii) On the interval $[0, 3]$, f has a critical value at $x = 1$, so check f's value here and at the endpoints:
   $f(0) = -1$, $f(3) = 17$, and $f(1) = -3$.
   Therefore the absolute maximum is 17 and the absolute minimum is -3.

   (b) (i) $f'(x) = 2x^{-1/3} - 1 = \dfrac{2 - x^{1/3}}{x^{1/3}}$.
   Therefore f has critical numbers $x = 0$ and $x = 8$.
   (ii) On the interval $[-1, 27]$ f has a critical value at $x = 0$ and $x = 8$, so check f's value here and at the endpoints:
   $f(-1) = 4$, $f(27) = 0$, $f(0) = 0$, and $f(8) = 4$.
   Therefore the absolute maximum is 4 and the absolute minimum is 0.

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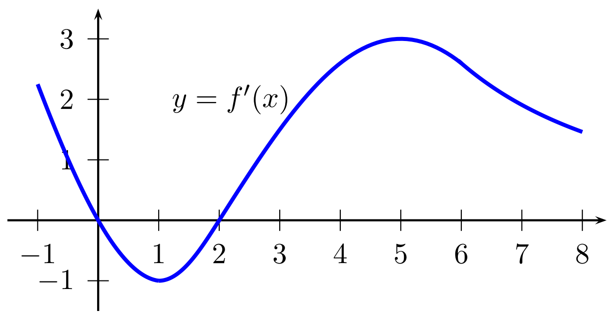
7. (10 pts) The graph of $f'(x)$ for the interval $[-1, 8]$ is shown below.



   (a) At what x-value(s) does f have a local maximum?
   (b) At what x-value(s) does f have a local minimum?
   (c) On what open interval(s) is f concave up?
   (d) On what open interval(s) is f concave down?
   (e) (Multiple Choice) If $f(4) = 5$, what does the Mean Value Theorem tell us would be the maximum value of $f(6)$? Choose the best answer.
   (i) 5 (ii) 7 (iii) 11 (iv) 18

   SOLUTION:

   (a) $x = 0$: Where f' goes from positive to negative.
   (b) $x = 2$: Where f' goes from negative to positive.
   (c) $(1, 5)$: Where f' is increasing.
   (d) $(-1, 1)$, $(5, 8)$: Where f' is decreasing.
   (e) (iii) 11: The maximum value of f' in the interval $(4, 6)$ is 3, so this is the maximum slope of the secant line from $(4, 5)$ to $(6, f(6))$, i.e., $f(6) \le f(4) + 3 \cdot (6 - 4) = 11$.

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8. (18 pts) Let $f(x) = \dfrac{-4}{x^2+3}$. Then $f'(x) = \dfrac{8x}{(x^2+3)^2}$ and $f''(x) = \dfrac{24(1-x^2)}{(x^2+3)^3}$.

Note: A possible answer to any of the following is "there are none".

(a) What are the equations of any horizontal asymptotes?

(b) What are the equations of any vertical asymptotes?

(c) On which open interval(s) is $f(x)$ increasing?

(d) On which open interval(s) is $f(x)$ decreasing?

(e) At what x-value(s) does f have a local minimum?

(f) At what x-value(s) does f have a local maximum?

(g) On which open interval(s) is $f(x)$ concave up?

(h) On which open interval(s) is $f(x)$ concave down?

(i) Find the $x-$coordinate of any inflection point.

SOLUTION:

sign analysis for f' and f'':

(a) $\lim_{x\to\pm\infty} \dfrac{-4}{x^2+3} = \lim_{x\to\pm\infty} \dfrac{0}{2x} \dfrac{1/x^2}{1/x^2} = 0$ so the only horizontal asymptote is $y = 0$.

(b) There are none. (Denominator is never zero.)

(c) $(0, \infty)$: See sign analysis of f' above.

(d) $(-\infty, 0)$: See sign analysis of f' above.

(e) $x = 0$: See sign analysis of f' above.

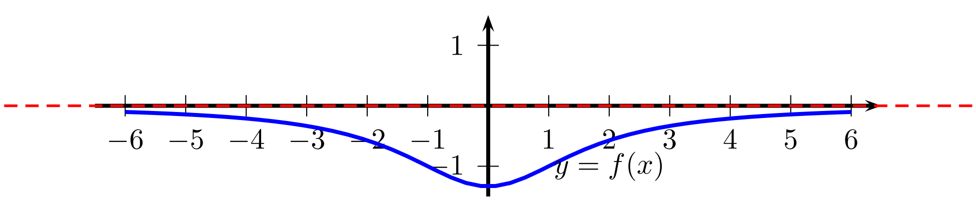
(f) none : See sign analysis of f' above.

(g) $(-1, 1)$: See sign analysis of f'' above.

(h) $(-\infty, -1)$, $(1, \infty)$: See sign analysis of f'' above.

(i) $x = 1$ and $x = -1$: See sign analysis of f'' above.

Here is the graph of f.



9. (4 pts) The conclusion of the Mean Value Theorem is that "there is some c in the interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$". What are the assumptions on f?

SOLUTION:

f should be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

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10. (6 pts) Show that $f(x) = 2x^3 + 3x + 1$ has exactly one root by completing the following steps:

    (a) Use the Intermediate Value Theorem to show that f has at least one root.

    (b) Use the Mean Value Theorem or Rolle's Theorem to show that f has exactly one root.

SOLUTION:

First, since f is a polynomial, it is continuous and differentiable everywhere and thus satisfies the hypotheses of all the referenced theorems.

(a) $f(-1) = -4$ and $f(0) = 1$ so by IVT there is at least one c in the interval $(-1, 0)$ such that $f(c) = 0$.

(b) Assume that f has two roots, say x_1 and x_2, then by either Rolle's Theorem or the MVT, there is some c in (x_1, x_2) such that $f'(c) = 0$. However since $f'(x) = 6x^2 + 3$ is always greater than zero, this produces a contradiction and there could not have been two roots to start with.
