KEY

Math 32
Calculus I
All sections

TUFTS UNIVERSITY
Department of Mathematics
Exam I

February 29, 2016
12-1:20 pm

No books, notes, **or calculators**. **TURN OFF YOUR CELL PHONE. ANYONE CAUGHT WITH THEIR CELL PHONE ON WILL BE GIVEN A 10 POINT DEDUCTION.** Cross out what you do not want us to grade. You **must** show work to receive full credit. Please try to write neatly. You need not simplify your answers unless asked to do so. You should evaluate standard trigonometric functions like $\tan(\pi/3)$. You are required to **sign** your exam book. With your signature, you pledge that you have neither given nor received assistance on this exam.

Problem	Point Value	Points
1	10	
2	6	
3	9	
4	12	
5	18	
6	12	
7	6	
8	6	
9	10	
10	6	
11	5	
	100	

1. (10 points) The Derivative

(a) State the definition of the derivative of a function $y = f(x)$ as a limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Use your definition to find the derivative of the function $f(x) = \dfrac{1}{5x+2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$(5x + 5h + 2)$

$$= \lim_{h \to 0} \frac{\frac{1}{5(x+h)+2} - \frac{1}{5x+2}}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{(5x+2) - (5(x+h)+2)}{(5(x+h)+2)(5x+2)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{5x+2 - 5x - 5h - 2}{(5(x+h)+2)(5x+2)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-5h}{(5(x+h)+2)(5x+2)}\right]$$

$$= \lim_{h \to 0} \frac{-5}{(5(x+h)+2)(5x+2)} = \frac{-5}{(5x+2)(5x+2)} = \frac{-5}{(5x+2)^2}$$

2. (6 points) Find the following limits or state that they do not exist. If you are able to describe a nonexistent limit by using the notation ∞ or $-\infty$ please do so. Make sure you show your work.

(a) $\lim_{x \to -3^-} \frac{2x}{(x+3)(x^2-9)}$ $\quad -6 \quad = +\infty$

$\downarrow$ 0 (negative) $\quad \searrow$ 0 (positive)

(b) $\lim_{x \to 0} \frac{\sin(4x)}{2x}$ $= \lim_{x \to 0} \frac{2\sin(4x)}{2 \cdot 2x} = \lim_{x \to 0} \frac{2\sin 4x}{4x}$

$= 2 \lim_{x \to 0} \frac{\sin 4x}{4x} = 2 \boxed{\lim_{t \to 0} \frac{\sin t}{t}} = 2 \cdot 1 = 2$

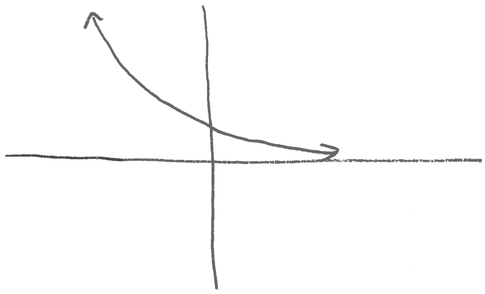
$\downarrow$

1

Let $t = 4x$. Then $t \to 0$ as $x \to 0$

3. (9 points) Find the following limits or state that they do not exist. If you are able to describe a nonexistent limit by using the notation ∞ or $-\infty$ please do so. Show your work. (In some cases a graph will suffice.)

(a) $\lim\limits_{x\to\infty} e^{-2x}$ $= 0$



(b) $\lim\limits_{x\to-\infty} \dfrac{14x^2+12}{10-x-x^2}$ $= \lim\limits_{x\to-\infty} \dfrac{14 + \overbrace{12/x^2}^{\to 0}}{\underbrace{10/x^2}_{\to 0} - \underbrace{1/x}_{\to 0} - 1} = -14$

divide each term by x^2

(c) Use the Squeeze Theorem to find $\lim\limits_{x\to\infty} \dfrac{\sin x+3}{x^2}$

Since $-1 \le \sin x \le 1$, for all x

$$\frac{-1+3}{x^2} \le \frac{\sin x + 3}{x^2} \le \frac{1+3}{x^2}$$

$$\frac{2}{x^2} \le \frac{\sin x + 3}{x^2} \le \frac{4}{x^2}$$

as $x \to \infty$ $\quad \to 0 \leftarrow$

So $\lim\limits_{x\to\infty} \dfrac{\sin x + 3}{x^2} = 0$

4. (12 points) Differentiate. DO NOT SIMPLIFY YOUR ANSWERS.

(a) $y = \sin^2(x)$ $= (\sin x)^2$

$y' = 2\sin x \cos x$

(b) $y = 10e^{3x}$

$y' = 10 \cdot 3 e^{3x}$ $(= 30 e^{3x})$

(c) $y = \ln(6 - x)$

$y' = \left(\frac{1}{6-x}\right)(-1)$

(d) $y = 2^x$

$y' = (\ln 2) 2^x$ or $y = e^{\ln 2^x} = e^{x \ln 2} = e^{(\ln 2)x}$

$y' = e^{(\ln 2)x} \cdot \ln 2$

5. (18 points) Differentiate. DO NOT SIMPLIFY YOUR ANSWERS.

(a) $y = \cos^6(e^{-2x})$ $= (\cos(e^{-2x}))^6$

$$y' = 6\cos^5(e^{-2x}) \bullet (-\sin(e^{-2x})) \bullet e^{-2x} \bullet -2$$

(b) $y = \dfrac{e^x + x^{2/3}}{\tan x}$

$$y' = \frac{(\tan x)(e^x + \tfrac{2}{3}x^{-1/3}) - (e^x + x^{2/3})(\sec^2 x)}{\tan^2 x}$$

(c) $y = x^{\cos x}$

Option 1: $y = e^{\ln x^{\cos x}}$

$$y = e^{\cos x \ln x}$$

$$y' = \left[e^{\cos x \ln x}\right] \cdot \left[(-\sin x)\ln x + (\cos x)\left(\tfrac{1}{x}\right)\right]$$

$$\left(= x^{\cos x}\left((-\sin x)(\ln x) + (\cos x)\left(\tfrac{1}{x}\right)\right)\right)$$

Option 2: Use logarithmic differentiation

$$y = x^{\cos x}$$

$$\ln y = \ln(x^{\cos x})$$

$$\ln y = (\cos x)(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = (-\sin x)(\ln x) + (\cos x)\left(\tfrac{1}{x}\right)$$

$$\frac{dy}{dx} = y\left[(-\sin x)(\ln x) + (\cos x)\cdot\left(\tfrac{1}{x}\right)\right]$$

$$\frac{dy}{dx} = x^{\cos x}\left[(-\sin x)(\ln x) + (\cos x)\left(\tfrac{1}{x}\right)\right]$$

6. (12 points) Differentiation. DO NOT SIMPLIFY YOUR ANSWERS.

(a) Find the second derivative of the function $y = e^{-5x}\cos x$

$$y' = -5e^{-5x}\cos x + e^{-5x}(-\sin x)$$

$$y'' = (25e^{-5x}\cos x + -5e^{-5x}(-\sin x)) + -5e^{-5x}(-\sin x) + e^{-5x}(-\cos x)$$

(b) Use logarithmic differentiation of find the derivative of $y = \dfrac{(x-2)^8(x+5)^{2/3}}{\sqrt{x-9}}$

$$\ln y = \ln\left[\frac{(x-2)^8(x+5)^{2/3}}{\sqrt{x-9}}\right]$$

$$\ln y = \ln (x-2)^8(x+5)^{2/3} - \ln (x-9)^{\frac{1}{2}}$$

$$= \ln(x-2)^8 + \ln(x+5)^{2/3} - \ln(x-9)^{\frac{1}{2}}$$

$$= 8\ln(x-2) + \frac{2}{3}\ln(x+5) - \frac{1}{2}\ln(x-9)$$

$$\frac{1}{y}\, dy/dx = \frac{8}{x-2} + \frac{2}{3}\left(\frac{1}{x+5}\right) - \frac{1}{2}\left(\frac{1}{x-9}\right)$$

$$dy/dx = y\left[\frac{8}{x-2} + \frac{2}{3}\left(\frac{1}{x+5}\right) - \frac{1}{2}\left(\frac{1}{x-9}\right)\right]$$

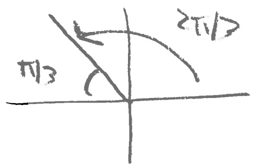
$$dy/dx = \left(\frac{(x-2)^8(x+5)^{2/3}}{\sqrt{x-9}}\right)\left[\frac{8}{x-2} + \frac{2}{3}\left(\frac{1}{x+5}\right) - \frac{1}{2}\left(\frac{1}{x-9}\right)\right]$$

7. (6 points) Find an equation for the line tangent to $f(x) = \sin(4x)$ at the point where x = $\frac{\pi}{6}$. You may leave your equation in point-slope form.

$f'(x) = \cos(4x) \cdot 4$

$f'(\pi/6) = 4\cos(4 \cdot \pi/6) = 4\cos 2\pi/3$

$f'(\pi/6) = 4\left(-\frac{1}{2}\right) = -2$



LINE: $y - f(\pi/6) = f'(\pi/6)(x - \pi/6)$

$y - \sin(4 \cdot \pi/6) = -2(x - \pi/6)$

$y - \sin(2\pi/3) = -2(x - \pi/6)$

$\boxed{y - \sqrt{3}/2 = -2(x - \pi/6)}$

8. (6 points) Find the slope of the line tangent to $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$.

Implicit differentiation

$4x^3 - (2xy + x^2(1 \cdot y')) + 4y^3 \cdot y' = 0$

$4(-1)^3 - (2(-1)(1) + (-1)^2 y') + 4(1)^3 \cdot y' = 0$

$-4 - (-2 + y') + 4y' = 0$

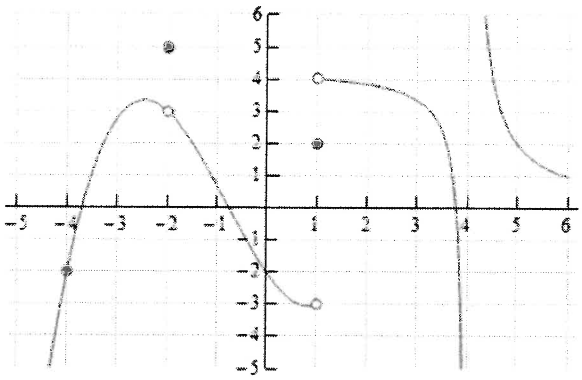
$-4 + 2 - y' + 4y' = 0$

$-2 + 3y' = 0$

$3y' = 2$

$y' = 2/3$

9. (10 points) Here is the graph of a function $y = f(x)$ drawn in the Cartesian plane.



Complete:

(a) i. $\lim_{x \to -2^-} f(x) = 3$

ii. $\lim_{x \to -2^+} f(x) = 3$

iii. $\lim_{x \to -2} f(x) = 3$

iv. Is $f(x)$ continuous at $x = -2$? Why or why not.

No - because $\lim_{x \to -2} f(x) = 3$ but $f(-2) = 5$ so

$\lim_{x \to -2} f(x) \neq f(-2)$

(b) i. $\lim_{x \to 1^-} f(x) = -3$

ii. $\lim_{x \to 1^+} f(x) = 4$

iii. $\lim_{x \to 1} f(x)$ does not exist

iv. Is $f(x)$ continuous at $x = 1$? Why or why not.

No - $\lim_{x \to 1} f(x)$ does not exist so it is not possible

for $\lim_{x \to 1} f(x) = f(1) = 2$

(c) List all intervals on which $f(x)$ is continuous.
(Assume the function is continuous for $x < -4$ and $x \geq 6$.

$(-\infty, -2)$, $(-2, 1)$, $(1, 4)$, $(4, +\infty)$

10. (6 points) Determine whether or not $f(x)$ is continuous at $x = 3$. Show all work.

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \text{ is} \\ 6 & \text{if } x = 3 \end{cases}$$

① $f(3) = 6$ so $f(3)$ is defined

② $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = \lim_{x \to 3} (x+3) = 6$

③ $f(3) = \lim_{x \to 3} f(x)$ so yes $f(x)$ is continuous at $x = 3$.

11. (5 points) Let $f(x) = -2x^3 + 6x + 3$. Then $f(x)$ is continuous for all real numbers x. Use the Intermediate Value Theorem to prove that the equation $-2x^3 + 6x + 3 = 0$ has a solution on the interval $[0, 2]$.

① Since $f(x)$ is continuous for all x, it is continuous on $[0,2]$.

② $f(0) = -2(0) + 6(0) + 3 = 3$

$f(2) = -2(2)^3 + 6(2) + 3 = -16 + 12 + 3 = -1$

③ Since $\underset{f(2)}{-1} < 0 < \underset{f(0)}{3}$, we know by the IVT that

there must be some number c, $0 < 2 < c$ such that

$f(c) = -2(c^3) + 6(2) + 3 = 0$.