Tufts University

Math 32 — Department of Mathematics — Spring 2014

Exam 1 Review Solutions

1.

(a) $\lim_{x\to -2} \frac{x^2+5x+6}{x+2} = \lim_{x\to -2} \frac{(x+2)(x+3)}{x+2} = \lim_{x\to -2} x+3 = \boxed{1}$

(b) $\lim_{x\to -\infty} \frac{2x-1}{\sqrt{6x^2+2}} = \lim_{x\to -\infty} \frac{x(2-1/x)}{\sqrt{x^2}(\sqrt{6+2/x^2})} = \lim_{x\to -\infty} \frac{x(2-1/x)}{|x|(\sqrt{6+2/x^2})}$

$\lim_{x\to -\infty} \frac{-2+1/x}{\sqrt{6+2/x^2}} = \boxed{-2/\sqrt{6}}$ where $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < -0 \end{cases}$ has been used.

(c) $\lim_{x\to -2} \frac{x^2+7x+10}{2x^2+5x+2} = \lim_{x\to -2} \frac{(x+5)(x+2)}{2(x^2+\frac{5}{2}x+1)} = \lim_{x\to -2} \frac{(x+5)(x+2)}{2(x+\frac{1}{2})(x+2)} =$

$\lim_{x\to -2} \frac{x+5}{2(x+\frac{1}{2})} = \frac{3}{2(\frac{-3}{2})} = \boxed{-1}$

(d) $|x+1| = \begin{cases} x+1, & x \geq -1 \\ -(x+1), & x < -1 \end{cases}$

$\lim_{x\to -1^-} \frac{|x+1|}{x+1} = \lim_{x\to -1^-} \frac{-(x+1)}{x+1} = -1$ and $\lim_{x\to -1^+} \frac{|x+1|}{x+1} = \lim_{x\to -1^+} \frac{x+1}{x+1} = 1.$

So $\lim_{x\to -1} \frac{|x+1|}{x+1}$ $\boxed{DNE}$

(e) $\lim_{x\to\infty} \frac{3x^2+2x+1}{5x^2-x} = \lim_{x\to\infty} \frac{x^2(3+2/x+1/x^2)}{x^2(5-1/x)} = \boxed{\frac{3}{5}}$

(f) $\lim_{x\to\pi} \cos^2 x = \cos^2 \pi = (-1)^2 = 1$

$y = \ln x$ is continuous at 1, so $\lim_{x\to\pi} \ln\cos^2 x = \ln\left(\lim_{x\to\pi}\cos^2 x\right) = \ln 1 = \boxed{0}$

(g) $\lim_{x\to\infty} \left(\sqrt{x^2+4x}-x\right)\left(\frac{\sqrt{x^2+4x}+x}{\sqrt{x^2+4x}+x}\right) = \lim_{x\to\infty} \frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x} =$

$\lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2}\left(\sqrt{1+\frac{4}{x^2}}+\frac{x}{\sqrt{x^2}}\right)} = \lim_{x\to\infty} \frac{4x}{|x|\left(\sqrt{1+\frac{4}{x^2}}+\frac{x}{|x|}\right)}$

$= \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{4}{x}}+1} = \boxed{2}$ where $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < -0 \end{cases}$ has been used.

(h) Must use the Squeeze Theorem since $\lim_{x\to\infty} \sin x$ DNE.

$2 \leq 3+\sin x \leq 4$ implies $-4x \leq -x(3+\sin x) \leq -2x$, so $e^{-4x} \leq e^{-x(3+\sin x)} \leq e^{-2x}$

Both $\lim_{x\to\infty} e^{-4x} = 0$ and $\lim_{x\to\infty} e^{-2x} = 0$, so by the Squeeze Theorem,

$\lim_{x\to\infty} e^{-x(3+\sin x)} = \boxed{0}$ as well.

(i) $\lim_{x\to 0} \frac{\sin^2(3x)}{x\cot(4x)} = \lim_{x\to 0} \left[3^2 \cdot 4 \cdot \frac{\sin(3x)}{3x} \cdot \frac{\sin(3x)}{3x} \cdot \frac{x^2}{\cos(4x)} \cdot \frac{\sin(4x)}{4x}\right] = 36 \cdot 1 \cdot 1 \cdot \frac{0}{1} \cdot 1 = \boxed{0}$

Note : $\lim_{z\to 0} \frac{\sin z}{z} = 1.$

2.

(a) $y = \dfrac{x^2 - \sqrt{x}}{3x + 2\sqrt{x}}$

$$\boxed{y' = \frac{(3x+2\sqrt{x})(2x - \frac{1}{2}x^{-\frac{1}{2}}) - (x^2 - \sqrt{x})(3 + 2\cdot\frac{1}{2}x^{-\frac{1}{2}})}{(3x+2\sqrt{x})^2}}$$

(b) $y = (4x^3 - 2\cos x + 5e^2)^{1/3}$ $\quad$ Note : $\frac{d}{dx}\cos x = -\sin x$

$$\boxed{y' = \frac{1}{3}(4x^3 - 2\cos x + 5e^2)^{-2/3}(12x^2 + 2\sin x)}$$

(c) $y = \sqrt{x}e^{2\tan x}$ $\quad$ Note : $\frac{d}{dx}\tan x = \sec^2 x$

$$\boxed{y' = \sqrt{x}(e^{2\tan x}(2\sec^2 x)) + e^{2\tan x}\cdot\frac{1}{2}x^{-\frac{1}{2}}}$$

(d) $y = e^{\sec x + 7x}$ $\quad$ Note : $\frac{d}{dx}\sec x = \sec x\tan x$

$$\boxed{y' = e^{\sec x + 7x}(\sec x\tan x + 7)}$$

(e) $y = \dfrac{1 - xe^x}{x + e^x}$

$$\boxed{y' = \frac{(x+e^x)(-xe^x + e^x(-1)) - (1 - xe^x)(1 + e^x)}{(x+e^x)^2}}$$

(f) Use implicit differentiation. Applying dy/dx to both sides of $y = e^y(2 + \sin x) - 2$,

$dy/dx = e^y\ dy/dx\ (2 + \sin x) + e^y \cdot \cos x$.

Plugging in $(x, y) = (0, 0)$, $dy/dx = 2dy/dx + 1$ so $\boxed{dy/dx = -1}$.

3.

(a) $\ln(4 - e^{-x}) = -x \qquad e^{\ln(4-e^{-x})} = e^{-x} \qquad 4 - e^{-x} = e^{-x}$

$2 = e^{-x} \qquad \ln 2 = -x$ so $\boxed{x = -\ln 2}$

(b) $0 = \ln(\sqrt{x} + x) + \ln(\sqrt{x} - x) = \ln((\sqrt{x}+x)(\sqrt{x}-x)) = \ln(x - x^2)$.
Exponentiating both sides: $1 = x - x^2$ or $x^2 - x + 1 = 0$.

Use quadratic formula: $\quad x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \dfrac{1 \pm \sqrt{-3}}{2}$.

$\boxed{\text{No solutions}}$ since $\sqrt{-3}$ does not give a real number.

4.

(a) $2\ln 6 + \ln\frac{3}{4} = \ln 36 + \ln\frac{3}{4} = \ln(36\cdot\frac{3}{4}) = \boxed{\ln(27)}$.

(b) $\ln\tan(\frac{4\pi}{3}) - \ln\sin(\frac{\pi}{6}) = \ln\sqrt{3} - \ln\frac{1}{2} = \boxed{\ln(2\sqrt{3})}$

(c) $\csc(-9\pi/4) = \csc(-2\pi - \pi/4) = \csc(-\pi/4) = \frac{1}{\sin(-\pi/4)} = \boxed{-\sqrt{2}}$

5.

(a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

(b) $f(x) = \sqrt{x^2+1}$

$$f'(x) = \lim_{x\to 0} \left(\frac{\sqrt{(x+h)^2-1}-\sqrt{x^2+1}}{h}\right)\left(\frac{\sqrt{(x+h)^2-1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2-1}+\sqrt{x^2+1}}\right)$$
$$= \lim_{h\to 0} \frac{(x+h)^2-(x^2+1)}{h(\sqrt{(x+h)^2-1}+\sqrt{x^2+1})} = \lim_{h\to 0} \frac{x^2+2xh+h^2+1-x^2-1}{h(\sqrt{(x+h)^2-1}+\sqrt{x^2+1})}$$
$$= \lim_{h\to 0} \frac{2xh+h^2}{h(\sqrt{(x+h)^2-1}+\sqrt{x^2+1})} = \lim_{h\to 0} \frac{2x+h}{(\sqrt{(x+h)^2-1}+\sqrt{x^2+1})} = \frac{2x}{2\sqrt{x^2+1}}$$
$$= \boxed{\frac{x}{\sqrt{x^2+1}}}$$

(c) $f(x) = \dfrac{1}{3x-1}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{3(x+h)-1}-\frac{1}{3x-1}}{h} = \lim_{h\to 0} \frac{3x-1-(3(x+h)-1)}{(3x-1)(3(x+h)-1)}\cdot\frac{1}{h}$$
$$= \frac{3x-1-3x-3h+1}{h(3x-1)(3(x+h)-1)} = \lim_{h\to 0} \frac{-3}{(3x-1)(3(x+h)-1} = \boxed{\frac{-3}{(3x-1)^2}}$$

6. $y = \dfrac{x^2-x-6}{x^3-9x} = \dfrac{(x-3)(x+2)}{x(x-3)(x+3)}$

$$\lim_{x\to\infty} \frac{x^2-x-6}{x^3-9} = \lim_{x\to\infty} \frac{x^2(1-\frac{1}{x}-\frac{6}{x^2})}{x^3(1-\frac{9}{x^2})} = 0$$

Similarly, $\lim_{x\to -\infty} \dfrac{x^2-x-6}{x^3-9x} = 0.$ So $\boxed{y=0}$ is a horizontal asymptote.

$$\lim_{x\to 0^-} \frac{(x-3)(x+2)}{x(x-3)(x+3)} = \lim_{x\to 0^-} \frac{x+2}{x(x+3)} = -\infty$$
$$\lim_{x\to 0^+} \frac{(x-3)(x+2)}{x(x-3)(x+3)} = \lim_{x\to 0^+} \frac{x+2}{x(x+3)} = \infty$$

So $\boxed{x=0}$ is a vertical asymptote.

$$\lim_{x\to -3^-} \frac{(x-3)(x+2)}{x(x-3)(x+3)} = \lim_{x\to -3^-} \frac{x+2}{x(x+3)} = -\infty$$
$$\lim_{x\to -3^+} \frac{(x-3)(x+2)}{x(x-3)(x+3)} = \lim_{x\to -3^+} \frac{x+2}{x(x+3)} = \infty$$

So $\boxed{x=-3}$ is a vertical asymptote.

$$\lim_{x\to 3} \frac{(x-3)(x+2)}{x(x-3)(x+3)} = \lim_{x\to 3} \frac{x+2}{x(x+3)} = \frac{5}{18}$$

$x=3$ is NOT a vertical asymptote.

7. $y = 2xe^x \qquad (0,0) \qquad y' = 2x \cdot e^x + e^x \cdot 2 \qquad m = y'(0) = 2(0)e^0 + e^0 \cdot 2 = 2$

   $y - 0 = 2(x - 0)$ So $\boxed{y = 2x}$ is the tangent line.

8. $\dfrac{1}{(1+x)^2} = \dfrac{1}{1+2x+x^2} \qquad y' = \dfrac{-2}{(1+x)^3}$

   The line $4y + x = 1 \qquad 4y = 1 - x \qquad y = \dfrac{1}{4} - \dfrac{x}{4}$ has slope $-\dfrac{1}{4}$

   We need to find x so that $\dfrac{-2}{(1+x)^3} = \dfrac{-1}{4}$

   $-8 = -(1+x)^3 \quad 8 = (1+x)^3 \quad 2 = 1 + x \quad x = 1$

   We want the line with $m = \dfrac{1}{4}$ that goes through $(1, y(1))$: $y(1) = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$

   $y - \dfrac{1}{4} = -\dfrac{1}{4}(x-1) \quad y = -\dfrac{x}{4} + \dfrac{1}{4} + \dfrac{1}{4} \quad \boxed{y = -\dfrac{x}{4} + \dfrac{1}{2}}$

9. Below are just partial solutions to (a)-(d) to help you assess your answers.

   (a) f is continuous on $[-2, 3) \cup (3, 7]$.

   f is discontinuous at 3 because it has a jump. $\lim\limits_{x \to 3^-} f(x) = 3$ but $\lim\limits_{x \to 3^+} f(x) = -1$

   (b) f is differentiable on $(-2, 1) \cup (1, 3) \cup (3, 5) \cup (5, 7)$.

   f is not differentiable at 1 or 5 because it has a corner at each of those x-coordinates.

   f is not differentiable at 3 because it is not continuous at 3.

   Functions are never differentiable at end points in their domains.

   (c) $f'(0) = 0$ horizontal tangent line.

   (d) $f'(4) = \dfrac{5}{2}$

   (e) and (f) should be discussed in class.

10. These are important problems to be able to do in preparation for the exam. They are mostly from the book, so you can check your answers in the back of the book or in the Solution manual on reserve in Tisch library for Math 32 and 34.