Proofs

Here are four 'proofs' of the statement "If a division is envy free, then it is fair." Can you find what is wrong with each of them?

1. Since the division is envy free, A in particular has no envy. Since A has no envy, the division must be fair to A. Similarly, B has no envy, so the division must be fair to B. This is true for all the participants, so it is fair to everyone.

2. Since the division is envy free, A does not want anyone else's slice. In other words, A got a largest slice (to A). This must be fair to A. B also thinks B got a largest slice, so it is fair to B. This is true for all the participants, so it is fair to everyone.

3. Proof by contrapositive. The contrapositive of the statement is "If a division is unfair, then there is envy." Since the division is unfair, every candidate gets less than $\frac{1}{N}$ of the cake. But if every candidate gets less than $\frac{1}{N}$, then the sums of the values of the slices is less than 1, a contradiction. So there must be envy.

4. Proof by contrapositive. Since the division is unfair, it is unfair to A. And since it is unfair to A, then A did not get the largest slice (in A's eyes). Therefore A envies whoever did.