MATH 19-01: PRACTICE PROBLEMS FOR FINAL EXAM

TUFTS UNIVERSITY DEPARTMENT OF MATHEMATICS
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Voting Systems

(1) *Refer to review materials for exams 1 and 2 on the course website. Use the Voting Handout for extra practice. (We covered everything on the handout except "polarizing candidates.")*

(2) *Consider the following preference schedule:*

×4	×3	×4	×2
B	A	C	C
A	B	A	B
C	C	B	A

Verify that A is a Condorcet candidate and find a voting method for which A is a losing spoiler.

A is a Condorcet candidate because A wins all the two-way consolidations: A beats B head-to-head (7-6) and beats C head-to-head (also 7-6).

Recall that the only ways for candidate X NOT to be a spoiler are

- $\mathcal{W} = \{X\}$ (X is the only winner);
- $\mathcal{W}' = \mathcal{W}$ (having X in or out changes nothing); or
- $\mathcal{W}' = \mathcal{W} \cup \{X\}$ (the *other* winners are the same with or without the presence of X).

If we use this schedule under the plurality method, $\mathcal{W} = \{C\}$ but $\mathcal{W}' = \{B\}$. This is because C has 6 first-place votes, which is the most in the original election, but if we consolidate A out of the preference schedule, that gives B seven first-place votes, which is enough to win. Thus A is a spoiler. Since $A \notin \mathcal{W}$, we furthermore conclude that A is a losing spoiler.

(3) *On a pairwise comparison graph with 55 voters, explain why all of the margins of victory must be odd.*

The margins of victory are obtained from 2-way consolidations. Consider two candidates, X and Y. Their head-to-head must look like this:

$\times r$	$\times s$
X	Y
Y	X

. Now if r voters prefer X and s voters prefer Y, we also know that the sum is $r + s = 55$, the total number of voters. That means that out of r and s, one must be even and the other is odd! (If they were both odd or both even, their sum would be even.) Finally, the margin of victory is the *difference* between the two numbers, and the difference between an even and an odd number is odd.

Apportionment

(4) *For apportionment, M denotes population and m denotes number of seats. The ith state or district has a quota $Q_i = \frac{M_i}{M} \cdot m$. Explain the terms in that formula and what the quota means.*

Q_i is the quota for the ith state, built as follows: M_i is the population of the ith state, so M_i/M is the proportion of the nation that lives in state i. So state i should fairly get the same proportion of the seats in Congress, which is $(M_i/M) \cdot m$. Therefore the quota represents this state's fair number of congressional seats.

Geometry of Gerrymandering

(5) *Recall that the compactness score of a shape is* $C(S) = \frac{400\pi A}{P^2}$. *Showing all work, verify that for a regular hexagon* H, *the score is* $C(H) = \frac{50\pi\sqrt{3}}{3}$.

This is solved completely on the HW9 solution set, posted on the course webpage. In number 4, the solution derives $C(S) = (400\pi 3\sqrt{3})/72$, which simplifies down to $(50\pi\sqrt{3})/3$.

(6) *The first Gingles factor requires that a minority population be "sufficiently large and geographically compact to constitute a majority in a single-member district." Ohio's population was 12.04% black on the 2010 census. How many congressional seats must it get in order that the black population can possibly pass this Gingles test? (For instance, if Ohio only gets one seat, then* 12.04% *is not a majority, so it would fail this Gingles factor.)*

The test asks that it is possible for a district to be drawn with the black population as a majority in that district.

If Ohio had two districts, for example, then each district would have 50% of Ohio's population, and 12.04 is not a majority of 50.

If Ohio had four districts, each district would have 25% of the population, and 12.04 is still not a majority of 25, but it's getting close!

If Ohio had five districts, then each one has 20% of the population, and 12.04 is a majority of that. So five is the smallest number of congressional seats needed so that the black population is sizable enough to satisfy the Gingles factor.

Probability and Elections

(7) *Consider a pairwise comparison graph with random arrows (no ties). Find the probability that there is a Condorcet candidate if the number of candidates is 3, 4, 5 and* n.

First, how many edges (connecting lines) does each graph have? For 3 candidates, the graph has 3 edges. For 4 candidates, there are 6 edges. For 5 candidates, there are 10 edges (it's a pentagon with a star inside). Finally, for n candidates, there are $\binom{n}{2}$ edges, because there's an edge for every pair of candidates that must go head-to-head, and there are $\binom{n}{2}$ ways to choose a pair out of the n. Some solutions to this problem will use that number, but let me give a solution that doesn't require that:

Let's look at the probability that a *particular* candidate is Condorcet. That means that all $n-1$ edges out of candidate A must have outward-pointing arrows. If in and out are equally likely, the probability of that is $(1/2)^{n-1}$, because it's 50-50 that the first one points out AND 50-50 for the next, and so on, so it's $\frac{1}{2} \cdot \frac{1}{2} \cdots$ multiplied $n-1$ times.

So to get the overall probability that any candidate is Condorcet, you need look at the events that (A is Condorcet) OR (B is Condorcet) OR (C is Condorcet), and so on for all the candidates. That means you'll add $(1/2)^{n-1}$ to itself n times.

$$\text{Prob(there is a Condorcet candidate)} = \frac{n}{2^{n-1}}.$$

For $n=3$, the probability is $3/4$; for $n=4$, it's $4/8 = 1/2$; and for $n=5$, it's $5/16$.

(8) *Suppose that we are going to hold a sequential election with 6 candidates,* M, A, T, H, S, C.

(a) How many sequences are possible?

There are $6! = 720$ ways to scramble (rearrange) 6 objects.

(b) If all sequences are equally likely, what is the probability that the sequence ends up in alphabetical order?

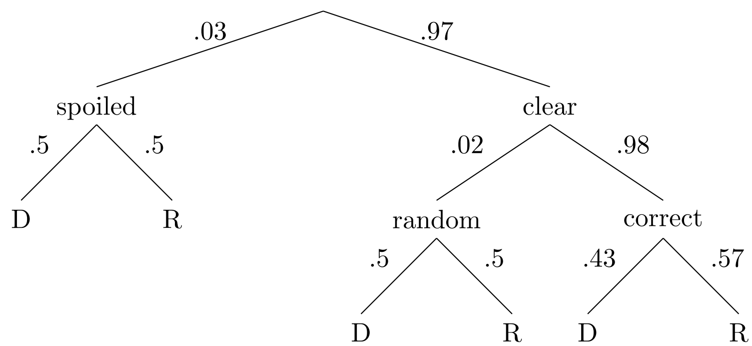
There's only one way to put them in alphabetical order (A, C, H, M, S, T) so the probability is $1/720$.

(c) What is the probability that M, A *are the first two candidates in the sequence?*

If those are the first two, in that order, then there are 4 spots left to fill with 4 candidates, which can happen in $4!$ ways. So the overall probability is $\frac{4!}{6!}$, which cancels down to $1/30$.

Here's a second way to get the same answer: For the first one in the sequence, if all six are equally likely, then the probability that it's M is 1/6. Now if the remaining 5 are equally likely to come second, the probability that it's A is 1/5. So the probability that BOTH things happen is $\frac{1}{6} \cdot \frac{1}{5} = \frac{1}{30}$.

(9) *In a particular election, 3% of ballots are spoiled. Let's suppose that election workers assign spoiled ballots randomly to D or R. Let's also suppose that out of the clear ballots, election workers have a 98% chance of tallying the ballot correctly and the remaining 2% of the time they assign it randomly to D or R. Finally, let's assume that the voter intent was for 43% to vote D and 57% to vote R. Make a decision tree showing all of the possible outcomes, and explain how to calculate the predicted vote totals for D and R in this model.*



To find the predicted proportion of D votes in the final tally, we just multiply down the branches that lead to D outcomes and add across the branches.

We get: $(.03)(.5)+(.97)(.02)(.5)+(.97)(.98)(.43) = .433...$ (rather than the 43% that would have occurred without any errors), and similarly the predicted R outcome is .566... (rather than 57%). So in this case, the spoiled ballots and tally errors have hurt the Republicans slightly and helped the Democrats.