We’ve defined the Isoperimetric Index (I.I.) of a shape to be the ratio \( P^2/A \), where \( P \) is the perimeter and \( A \) is the area of the shape.

Now for any shape \( S \), let’s define its compactness score 
\[
C(S) = \frac{400\pi A}{P^2}.
\]

(1) (a) Verify that the I.I. of a circle of radius 10 is the same as for a circle of radius 3.

Going further, verify that the I.I. of a circle of radius \( r \) does not depend on \( r \).

Area of a circle: \( \pi r^2 \). Perimeter of a circle: \( 2\pi r \). And \( II(S) = P^2/A \).

For circle of radius 10, \( A = 100\pi \) and \( P = 20\pi \), so \( P^2 = 400\pi^2 \). Thus \( II = 400\pi^2/100\pi = 4\pi \).

For a circle of radius 3, \( A = 9\pi \) and \( P = 6\pi \), so \( P^2 = 36\pi^2 \). Thus \( II = 36\pi^2/9\pi = 4\pi \).

And in general, \( II(\Box) = (4\pi^2 r^2)/(\pi r^2) = 4\pi \).

(b) Verify that the I.I. of a square with a side of length \( s \) does not depend on \( s \).

For a square with side \( s \), we have \( A = s^2 \) and \( P = 4s \implies P^2 = 16s^2 \), so \( II(\Box) = P^2/A = 16s^2/s^2 = 16 \).

(c) Suppose that a rectangle has length \( \ell \) and width \( w \). Show that the I.I. only depends on the ratio \( \ell/w \).

In this case, \( A = \ell w \) and \( P = 2\ell + 2w \), so \( P^2 = 4(\ell^2 + 2\ell w + w^2) \). Then
\[
II(\text{rect}) = 4\frac{\ell^2 + 2\ell w + w^2}{\ell w} = 4\left(\frac{\ell}{w} + 2 + \frac{w}{\ell}\right) = 4\left(\frac{\ell}{w} + 2 + \frac{1}{\ell/w}\right).
\]

(2) First show that the \( C(S) \) formula is equivalent to \( \frac{100 \cdot II(\text{circle})}{II(S)} \).

Now, using the isoperimetric theorem, prove that \( 0 \leq C(S) \leq 100 \) for all shapes \( S \).

Well, we saw above that \( II(\text{circle}) = 4\pi \) for any circle. Thus
\[
C(S) = \frac{400\pi A}{P^2} = \frac{100 A \cdot II(\text{circle})}{P^2} = 100 \cdot II(\text{circle}) \cdot \frac{A}{P^2} = \frac{100 \cdot II(\text{circle})}{P^2/A} = \frac{100 \cdot II(\text{circle})}{II(S)}
\]

Now, the Isoperimetric Theorem says that \( II(\text{circle}) \leq II(S) \) for any shape \( S \), or in other words, \( II(\text{circle})/II(S) \leq 1 \). But then substituting in to the \( C(S) \) formula immediately gives \( C(S) \leq 100 \).

In the other direction, \( C(S) \) is clearly non-negative because all of its terms are non-negative. And note that it can be arbitrarily close to zero: for instance, we’ll see in the next part that making a very skinny rectangle (with \( \ell/w \) very large) will cause \( C(S) \) to be as small as we want.

(3) Find the compactness scores of the shapes from the last part (circle of radius \( r \), square of side \( s \), and rectangle of sides \( \ell \) and \( w \)).

Now we know we can just compute \( C(S) = 400\pi/II(S) \).

\begin{align*}
II(\text{circle}) &= 4\pi \implies C(\text{circle}) = 400\pi/4\pi = 100. \\
II(\text{square}) &= 16 \implies C(\text{square}) = 400\pi/16 = 25\pi \approx 78.5. \\
II(\text{rectangle}) &= 4\left(\frac{\ell}{w} + 2 + \frac{1}{\ell/w}\right) \implies C(\text{rectangle}) = \frac{100\pi}{2 + \frac{1}{\ell/w}}.
\end{align*}

For instance, if \( R \) is a \( 10 \times 1 \) rectangle, then \( C(R) = \frac{100\pi}{2 + \frac{1}{10}} \approx 25.9 \).
(4) Let $H$ be a regular hexagon, and let $H'$ be a hexagon with vertices $(1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1)$. Let $O$ be a regular octagon and $O'$ an octagon with vertices $(2,1), (1,2), (-1,2), (-2,1), (-2,-1), (1,-2), (2,-1)$. Sketch these shapes, find their compactness scores, and make a conjecture about which polygons are the most “compact.”

Let’s start with the hexagon $H$. Note that we can pick any scale we want, because compactness scores are independent of scale, so we’ll give it side length 1.

$$H = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\fill (0,0) -- (0.5,0) -- (0.5,0.5) -- cycle;
\end{tikzpicture}
\end{array}$$

Since the area of the hexagon is six times as much as one of these triangles, we get $A(H) = 3\sqrt{3}/2$. On the other hand, the perimeter is $P = 6$, so we have $P^2 = 36$. Therefore we get $C(H) = (400\pi \cdot 3\sqrt{3})/72 = 90.689...$

How about the non-regular hexagon $H'$? This is made up of two squares with side 1 (since we’re picking the scale, again) and two right triangles.

$$H' = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\fill (0,0) -- (0.5,0) -- (0.5,0.5) -- cycle;
\end{tikzpicture}
\end{array}$$

This gives a perimeter of $P = 4 + 2\sqrt{2}$ and an area of $A = 3$. Now $P^2 = (4 + 2\sqrt{2})^2 = 16 + 16\sqrt{2} + 8 = 24 + 16\sqrt{2}$, so $C(H') = \frac{1200\pi \cdot 24 + 16\sqrt{2}}{96 + 64\sqrt{2}} = 80.85...$

We can handle the regular octagon by noticing that it is a square with some missing triangular chips. Those chips have area $\frac{1}{2}bh = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, and if we filled them in we’d have a square with side length $s = 1 + \frac{2}{\sqrt{2}} = 1 + \sqrt{2}$ and area $A = s^2 = 3 + 2\sqrt{2}$, so the octagon itself has area one less than that, or $A(O) = 2 + 2\sqrt{2}$. And its perimeter is 8, so we get $C(O) = \frac{400\pi \cdot (2+2\sqrt{2})}{64} = 94.80...$

And finally, we have the non-regular octagon.

$$O' = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\fill (0,0) -- (0.5,0) -- (0.5,0.5) -- cycle;
\end{tikzpicture}
\end{array}$$

So the square would have been 4 units on each side for an area of 16, but we have to subtract four chips of area 1/2, so we get $A(O') = 14$. On the other hand $P = 8 + 4\sqrt{2}$, so we have $C(O') = \frac{400\pi \cdot 14}{96 + 64\sqrt{2}} = 94.32...$

All together we have $C(H') < C(H) < C(O') < C(O)$. This suggests that regular $n$-gons are the most compact among $n$-gons for each $n$. Now it certainly can’t be true that simply having more sides makes you more compact (consider a long skinny rectangle, which is certainly less compact than an equilateral triangle even though it has more sides!), but we can also conjecture that among the regular polygons, having more sides makes you more like a circle and therefore more compact.
(5) The last two problems asked you to estimate the compactness scores of some actual voting districts. Here are a few scans of nice student solutions!
Students’ estimates for Gerry’s Salamander: $C = 7.43, 8.62, 16.75, 21.17$
My estimate: $C \approx 10 \quad \text{(Note: I don’t think the “wings” are included.)}$

Students’ estimates for TX-25: $C = 2.94, 5.07$
My estimate: $C \approx 5$

Historical note: in Bush v. Vera (1996), several majority-minority districts were invalidated by the court, while other majority-white districts were left intact. This map of TX-25 was used by Justice Stevens to point out that this treatment was inconsistent—this atrociously non-compact shape is one of the majority-white districts that was left in place by the court!