MATH 19-01: HW 9 SOLUTIONS

We've defined the Isoperimetric Index (I.I.) of a shape to be the ratio P^2/A, where P is the perimeter and A is the area of the shape.

Now for any shape S, let's define its *compactness score* $C(S) = \dfrac{400\pi A}{P^2}$.

(1) (a) *Verify that the I.I. of a circle of radius 10 is the same as for a circle of radius 3. Going further, verify that the I.I. of a circle of radius r does not depend on r.*

Area of a circle: πr^2. Perimeter of a circle: $2\pi r$. And $II(S) = P^2/A$.
For circle of radius 10, $A = 100\pi$ and $P = 20\pi$, so $P^2 = 400\pi^2$. Thus $II = 400\pi^2/100\pi = 4\pi$.
For a circle of radius 3, $A = 9\pi$ and $P = 6\pi$, so $P^2 = 36\pi^2$. Thus $II = 36\pi^2/9\pi = 4\pi$.
And in general, $II(\bigcirc) = (4\pi^2 r^2)/(\pi r^2) = 4\pi$.

(b) *Verify that the I.I. of a square with a side of length s does not depend on s.*

For a square with side s, we have $A = s^2$ and $P = 4s \implies P^2 = 16s^2$, so $II(\square) = P^2/A = 16s^2/s^2 = 16$.

(c) *Suppose that a rectangle has length ℓ and width w. Show that the I.I. only depends on the ratio ℓ/w.*

In this case, $A = \ell w$ and $P = 2\ell + 2w$, so $P^2 = 4(\ell^2 + 2\ell w + w^2)$. Then

$$II(\text{rect}) = 4\frac{\ell^2 + 2\ell w + w^2}{\ell w} = 4\left(\frac{\ell}{w} + 2 + \frac{w}{\ell}\right) = 4\left(\frac{\ell}{w} + 2 + \frac{1}{\ell/w}\right).$$

(2) *First show that the $C(S)$ formula is equivalent to $\dfrac{100 \cdot II(\text{circle})}{II(S)}$.*
Now, using the isoperimetric theorem, prove that $0 \le C(S) \le 100$ for all shapes S.

Well, we saw above that $II(\text{circle}) = 4\pi$ for any circle. Thus

$$C(S) = \frac{400\pi A}{P^2} = \frac{100A \cdot II(\text{circle})}{P^2} = 100 \cdot II(\text{circle}) \cdot \frac{A}{P^2} = \frac{100 \cdot II(\text{circle})}{P^2/A} = \frac{100 \cdot II(\text{circle})}{II(S)}$$

Now, the Isoperimetric Theorem says that $II(\text{circle}) \le II(S)$ for any shape S, or in other words, $II(\text{circle})/II(S) \le 1$. But then substituting in to the $C(S)$ formula immediately gives $C(S) \le 100$.

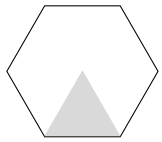
In the other direction, $C(S)$ is clearly non-negative because all of its terms are non-negative. And note that it can be arbitrarily close to zero: for instance, we'll see in the next part that making a very skinny rectangle (with ℓ/w very large) will cause $C(S)$ to be as small as we want.

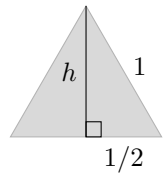
(3) *Find the compactness scores of the shapes from the last part (circle of radius r, square of side s, and rectangle of sides ℓ and w).*

Now we know we can just compute $C(S) = 400\pi/II(S)$.
$II(\text{circle}) = 4\pi \implies C(\text{circle}) = 400\pi/4\pi = 100$.
$II(\text{square}) = 16 \implies C(\text{square}) = 400\pi/16 = 25\pi \approx 78.5$.
$II(\text{rectangle}) = 4\left(\frac{\ell}{w} + 2 + \frac{1}{\ell/w}\right) \implies C(\text{rectangle}) = \dfrac{100\pi}{\frac{\ell}{w} + 2 + \frac{1}{\ell/w}}$.
For instance, if R is a 10×1 rectangle, then $C(R) = \frac{100\pi}{12.1} \approx 25.9$.

(4) *Let H be a regular hexagon, and let H' be a hexagon with vertices* $(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)$. *Let O a regular octagon and O' an octagon with vertices* $(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2)$, $(1,-2),(2,-1)$. *Sketch these shapes, find their compactness scores, and make a conjecture about which polygons are the most "compact."*

Let's start with the hexagon H. Note that we can pick any scale we want, because compactness scores are independent of scale, so we'll give it side length 1.

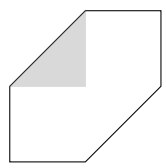
$H =$ 

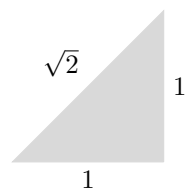


$$(1/2)^2 + h^2 = 1^2$$
$$h^2 = 3/4 \implies h = \sqrt{3}/2$$
$$A(\text{shaded triangle}) = \tfrac{1}{2}bh = \tfrac{\sqrt{3}}{4}$$

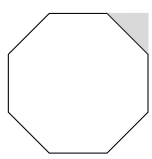
Since the area of the hexagon is six times as much as one of these triangles, we get $A(H) = \frac{3\sqrt{3}}{2}$. On the other hand, the perimeter is $P = 6$, so we have $P^2 = 36$. Therefore we get $C(H) = (400\pi \cdot 3\sqrt{3})/72 = 90.689...$

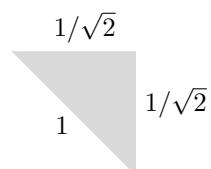
How about the non-regular hexagon H'? This is made up of two squares with side 1 (since we're picking the scale, again) and two right triangles.

$H' =$ 



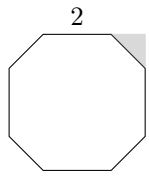
This gives a perimeter of $P = 4 + 2\sqrt{2}$ and an area of $A = 3$. Now $P^2 = (4+2\sqrt{2})^2 = 16 + 16\sqrt{2} + 8 = 24 + 16\sqrt{2}$, so $C(H') = \frac{1200\pi}{24+16\sqrt{2}} = 80.85...$

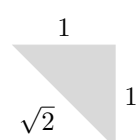
$O =$ 



We can handle the regular octagon by noticing that it is a square with some missing triangular chips. Those chips have area $\frac{1}{2}bh = \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{4}$, and if we filled them in we'd have a square with side length $s = 1 + \frac{2}{\sqrt{2}} = 1 + \sqrt{2}$ and area $A = s^2 = 3 + 2\sqrt{2}$, so the octagon itself has area one less than that, or $A(O) = 2 + 2\sqrt{2}$. And its perimeter is 8, so we get $C(O) = \frac{400\pi \cdot (2+2\sqrt{2})}{64} = 94.80...$

And finally, we have the non-regular octagon.

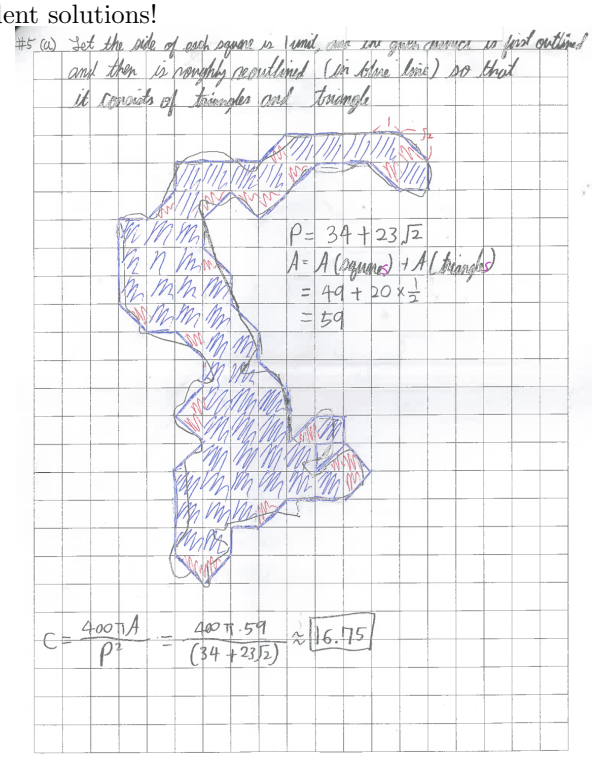
$O' =$ 

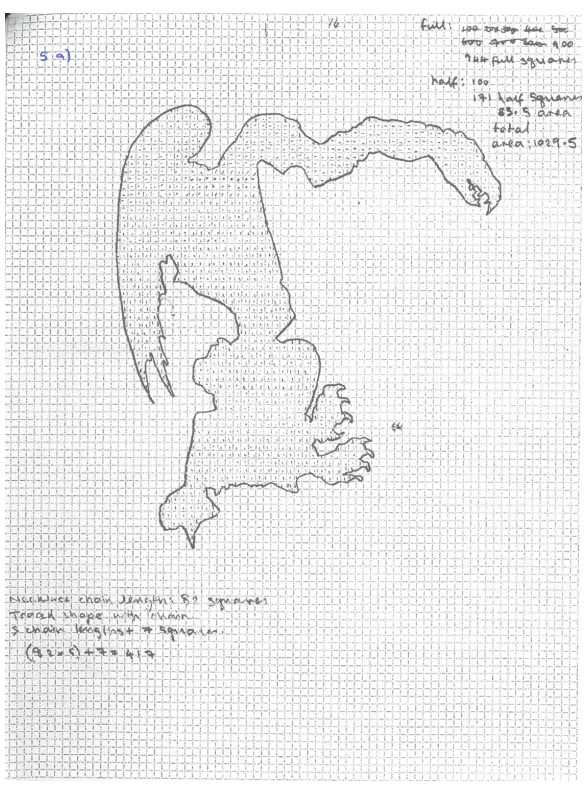


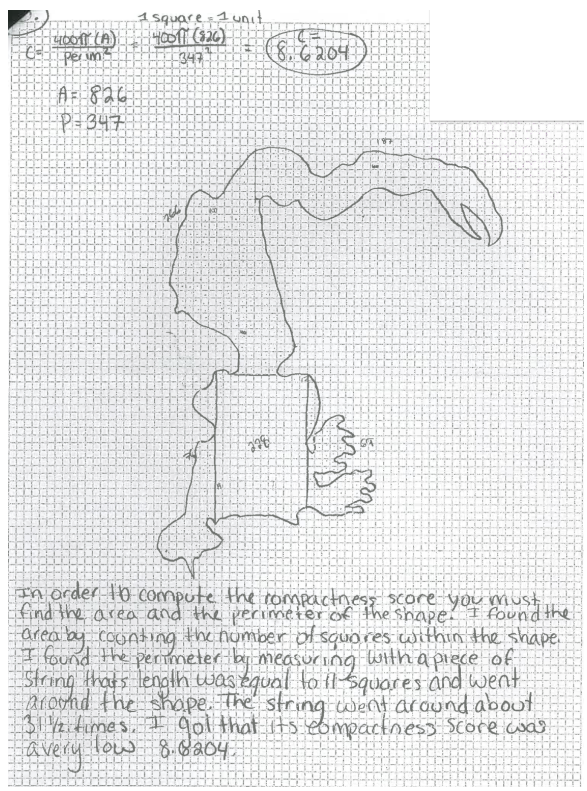
So the square would have been 4 units on each side for an area of 16, but we have to subtract four chips of area 1/2, so we get $A(O') = 14$. On the other hand $P = 8 + 4\sqrt{2}$, so we have $C(O') = \frac{400\pi \cdot 14}{96+64\sqrt{2}} = 94.32...$

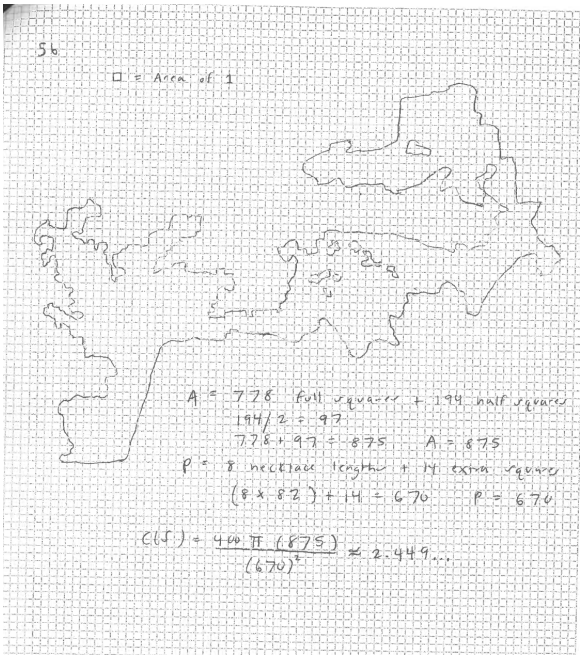
All together we have $C(H') < C(H) < C(O') < C(O)$. This suggests that regular n-gons are the most compact among n-gons for each n. Now it certainly can't be true that simply having more sides makes you more compact (consider a long skinny rectangle, which is certainly less compact than an equilateral triangle even though it has more sides!), but we can also conjecture that among the *regular* polygons, having more sides makes you more like a circle and therefore more compact.

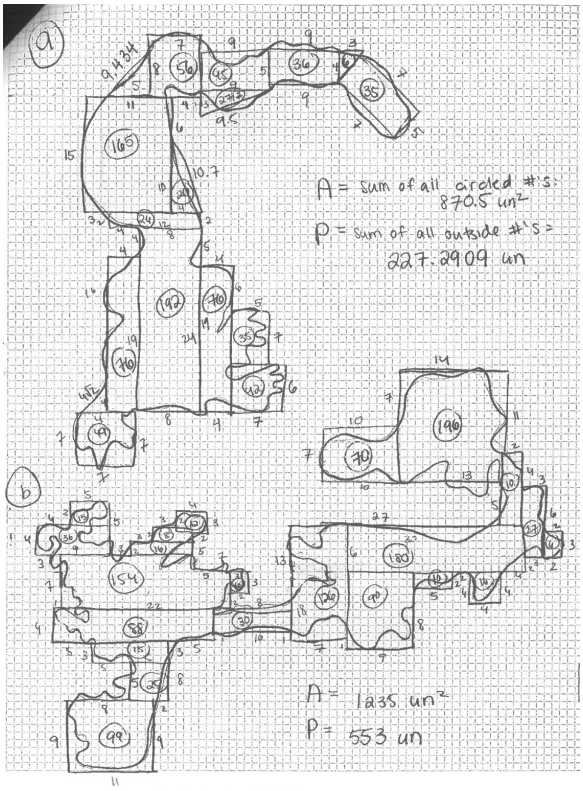
(5) The last two problems asked you to estimate the compactness scores of some actual voting districts. Here are a few scans of nice student solutions!











Students' estimates for Gerry's Salamander: $C = 7.43, 8.62, 16.75, 21.17$
My estimate: $C \approx 10$ (Note: I don't think the "wings" are included.)

Students' estimates for TX-25: $C = 2.94, 5.07$
My estimate: $C \approx 5$

Historical note: in Bush v. Vera (1996), several majority-minority districts were invalidated by the court, while other majority-white districts were left intact. This map of TX-25 was used by Justice Stevens to point out that this treatment was inconsistent—this atrociously non-compact shape is one of the majority-white districts that was left in place by the court!