MATH 19-01: HW 7

(1) Show that Dictatorship is a Pareto-efficient and strongly monotonic single-winner system. (In other words, it satisfies the hypotheses of the Müller-Satterthwaite theorem.) Note there are three separate things to verify here.

Claim 1: Dictatorship is single-winner.

Proof: In the Dictatorship system, the Dictator's 1st choice is the sole winner. □

Claim 2: Dictatorship is Pareto-efficient.

Proof: The Dictator is one of the voters, so if every voter ranks X first, then the Dictator does too, making X the sole winner. □

Claim 3: Dictatorship is strongly monotonic.

Proof: Suppose for a particular pref. schedule and choice of Dictator, that $W_{dict} = \{A\}$. That means A is the Dictator's 1st choice.

Now any move favorable to A keeps that true, so it can't change A from a winner to a loser.

A move neutral to A keeps them in the same position in every column, so the Dictator still ranks them 1st and $W' = \{A\}$.

The last thing to check is for some other candidate, say D, could a move neutral to D turn D into a winner?

But no! If D doesn't move in the columns, then they don't become the Dictator's 1st choice, so $D \notin W'$.

I've shown that favorable moves can't change a winner to a loser and neutral moves can't change $W \to L$ or $L \to W$, as needed. □

(2) *This is an opinion question: your answer can be anything as long as you explain your reasoning.* How reasonable is it to insist that a voting system be single-winner? Does your answer change if the number of candidates (n) is high or low? Does it change if the number of voters (N) is high or low?

Not very reasonable! What about

#1	#2
A	B
B	A

There's no good way to break a tie.

If the number of candidates is high, it seems (to me) even likelier that two could have a good case to win, so single-winner is maybe even <u>more</u> <u>unreasonable</u>.

However, if the number of voters is high, then exact ties become less likely (in some sense) so it's possibly a little <u>more</u> <u>reasonable</u> to have just one winner.

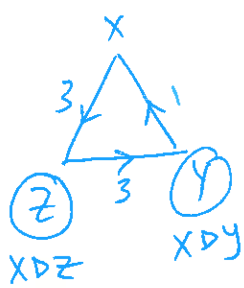
(3) Consider the following preference schedules:

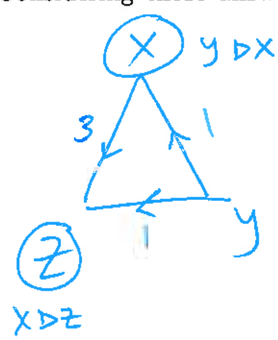
×3	×2	×2
X	Y	Z
Z	X	Y
Y	Z	X

$\xrightarrow[\text{to } X]{\text{neutral}}$

×3	×2	×2
X	Y	Y
Z	X	Z
Y	Z	X

Who wins each one, by the beatpath method? Considering those answers, does that tell you whether beatpath is strongly monotonic?





$W_{bp} = \{X\}$ $\qquad$ $W'_{bp} = \{Y\}$

The move was <u>neutral to X</u> but switched X from winner to loser,

so Beatpath is <u>not</u> strongly monotonic.

(4) (a) Following the proof of Müller-Satterthwaite, suppose you know you're working with an unknown single-winner voting system that is Pareto-efficient and strongly monotonic. Narrate the proof using the following sequence of detailed preference schedules.

①

#1	#2	#3	#4	#5
P	P	P	P	P
M	M	M	M	M
N	N	N	N	N
O	O	O	O	O

W=P by PE

②

#1	#2	#3	#4	#5
P	P	P	P	P
O	M	M	M	M
M	N	N	N	N
N	O	O	O	O

W=P by PE

③

#1	#2	#3	#4	#5
O	P	P	P	P
P	M	M	M	M
M	N	N	N	N
N	O	O	O	O

W ≠ M, N by unanimity (P > M, N)
So W=O or W=P

(W=O or P)

④ k=4

#1	#2	#3	#4	#5
O	O	O	P	P
P	P	P	O	M
M	M	M	M	N
N	N	N	N	O

⑤ k=4

#1	#2	#3	#4	#5
O	O	O	O	P
P	P	P	P	M
M	M	M	M	N
N	N	N	N	O

W=P ← (by choice of k) → W=O

neut to O

⑥

#1	#2	#3	#4	#5
O	O	O	O	M
M	M	M	P	N
N	N	N	M	P
P	P	P	N	O

W=O by SM from ⑤

neut O → P

neutral to M, N

⑦

#1	#2	#3	#4	#5
O	O	O	P	M
M	M	M	O	N
N	N	N	M	P
P	P	P	N	O

W ≠ M, N by SM from ⑥
W ≠ O by SM from ④
∴ W=P

neut to P

⑧

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	P
P	P	P	O	O

W=P by SM from ⑦

neut to M, N

⑨

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	O
P	P	P	O	P

W ≠ O by unanimity (M, N > O)
W ≠ M, N by SM from ⑧
∴ W=P

(b) If we're showing that the unknown voting system is Dictatorship of the kth voter using this proof technique, what is the value of k in the example above?

$k=4$

(That's where the winner "flipped" from P to O as we proceeded left to right moving O first into second place, then first.)

(c) The point of getting to a "pathological" preference schedule like 9 is that from there you can get to ANY detailed schedule in which voter k ranks candidate P first with a combination of moves neutral to and favorable to P.

Check this by filling in a preference schedule in the middle where the transitions are as described here.

9

#1	#2	#3	#4	#5
M	M	M	P	M
N	N	N	M	N
O	O	O	N	O
P	P	P	O	P

$\xrightarrow[\text{to } P]{\text{neut}}$

10

#1	#2	#3	#4	#5
O	O	N	P	O
M	M	O	O	M
N	N	M	N	N
P	P	P	M	P

$\xrightarrow[\text{to } P]{\text{fav}}$

11

#1	#2	#3	#4	#5
P	O	N	P	O
O	P	O	O	P
M	M	M	N	M
N	N	P	M	N

Who wins in that final schedule and why?

By SM, since W=P in 9 and the moves were neutral or favorable to P, we must still have W=P in 11.

(d) Note that 11 could have been *anything at all* as long as voter k liked P best! Explain why this finally proves that voter k is a "Dictator."

The def. of a Dictator is: if they like a candidate best, the candidate wins!

k can make P win just by ranking them first.

Nothing was special about P at the outset, so every candidate similarly has a voter who makes them win anytime that voter ranks them first.

Suppose ~~voter~~ voter 4 is the special voter for candidate P but voter 2 is the special voter for candidate N.

That's impossible! ... Because

O	N	O	P	O
N	O	N	O	N
P	P	P	N	P
M	M	M	M	M

would cause a contradiction.

So the process picks out the same Dictator, no matter which candidate we focus on. □