Redistricting and the Isoperimetric Problem
Math 19-03 Lecture 23

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23 April 2015
Announcements

- Exam: Monday May 4th, 7:00-10:00pm
- Office hours next week: Monday 7:30-9:00pm, Wednesday 10:30am-12:00pm
- Bonus rounds:
  1. Thursday April 30th, TBD, 10:30am-12:00pm (check course webpage for details)
  2. Sunday May 3rd, Robinson 152, 7:00-8:30pm
Today

1. The Isoperimetric Problem
2. Districting
3. Isoperimetric Problems Today
4. Review
The story of Dido (∼ 825 BCE)

Dido’s question: How much land can be bound by a bull’s hide?

Source: Ashbaugh & Benguria
The Isoperimetric Problem: how to maximize area with a fixed perimeter?
Rephrased: how to minimize perimeter with a fixed area?

Example. Isoperimetric question for rectangles.

\[
P = 4, \ A = 1 \quad P = 5, \ A = 1 \quad P = 6\frac{2}{3}, \ A = 1
\]
Fact. The square solves the isoperimetric problem for rectangles: minimum perimeter per unit area.

Theorem. (Zenodorus, ∼200 BCE) A regular polygon with \( n \) sides solves the isoperimetric problem for \( n \)-sided polygons.
Is there a shape which maximizes an area $A$ for a given perimeter $L$?

**The Isoperimetric Theorem** (Steiner, 1842).

Yes! It’s the circle of circumference $L$

**The Isoperimetric Inequality.** $L^2 - 4\pi A \geq 0$, with equality only for the circle.
Detail from a sketch made in commemoration of Carlos Quintos’ campaign on the doubled-walled city of Tunis, clearly satisfying the isoperimetric property of the circle. (31 August 1535). Source: Ashbaugh & Benguria
City of Boston Demographic Information: The red dots show white people, blue is black, orange is Hispanic, green is Asian, and yellow is other, according to maps of 2010 Census data by Eric Fischer.
Source: Business Insider
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Source: Business Insider
The original Gerry-Mander of the Jefferson Party in 1812.
Source: SIAM news
The Isoperimetric Problem
Districting
Isoperimetric Problems Today
Review

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S. Bray
25 neighborhoods, 5 districts equal size, 1 rep each district.

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- # W = 12 → 2.4 reps
- # B = 9 → 1.8 reps
- # H = 4 → .8 reps
The Isoperimetric Problem
Districting
Isoperimetric Problems Today
Review

S. Bray

- W: 4 reps
- B: 1 rep
- H: 0 reps
### The Isoperimetric Problem

#### Districting

### Isoperimetric Problems Today

**Review**

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- **W**: 1 full, 2 half → 2 reps
- **B**: 2 full, 1 half → 2.5 reps
- **H**: 0 full, 1 half → .5 reps
The Isoperimetric Problem

Districting

Isoperimetric Problems Today

Review

W: 2 reps
B: 2 reps
H: 1 rep
Optimal isoperimetric for district lines: how can we minimize total perimeter of the five districts?

\[ P = 12 \]
Optimal isoperimetric for district lines: how can we minimize total perimeter of the five districts?

\[ P = 12 \]
Optimal isoperimetric for district lines: how can we minimize total perimeter of the five districts?

- Spread: $P = 12$
- Compact: $P = 10$

An isoperimetrically optimal districting scheme will have more compact type districts and fewer spread type districts.

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Optimal isoperimetric for district lines: how can we minimize total perimeter of the five districts?

$P = 12$

“spread”

$P = 10$

“compact”

An isoperimetrically efficient districting scheme will have more compact type districts and fewer spread type districts.
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- W: 4 reps
- B: 1 rep
- H: 0 reps

# spread = 3
# compact = 2
Inefficient
The Isoperimetric Problem
Districting
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- **W**: 1 full, 2 half → 2 reps
- **B**: 2 full, 1 half → 2.5 reps
- **H**: 0 full, 1 half → .5 reps

# spread = 4
# compact = 1
Inefficient
The Isoperimetric Problem
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W: 2 reps
B: 2 reps
H: 1 rep

# spread = 1
# compact = 4
Efficient

Optimal !!!
Computer scientists study the isoperimetric problem for districting in the United States: *Washington Post*

My officemate digs it, for his research about math: *Exploring Isoperimetric Inequalities in Heisenburg space.*
Review time!

Resources on course webpage