Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices must be turned off and put away during the exam. Unless otherwise stated, you must show all work to receive full credit. Please reference $\vec{0}$ with the vector space to which it belongs, e.g. the zero vector in $V$ should be subscripted $O_V$. You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.

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1. (10 points)

Let

\[
A = \begin{pmatrix}
1 & 1 & 4 & 1 & 2 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
1 & -1 & 0 & 0 & 2 \\
2 & 1 & 6 & 0 & 1
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 & -1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then \(A\) and \(B\) are row equivalent. (You do not need to show this.)

(a) Find a basis \(B\) for \(\text{Col} \ A\).

(b) Show that \(\text{Col} \ A \neq \text{Col} \ B\) by finding a vector \(v\) that is in \(\text{Col} \ A\) but not \(\text{Col} \ B\). Briefly justify.

(c) Find the dimension of \(\text{Nul} \ A\).
2. (12 pts) For this whole problem let $V$ be a vector space with basis $B = \{b_1, b_2\}$.

(a) $V$ is isomorphic to $\mathbb{R}^n$ where $n = \phantom{0}$.

(b) Using the $n$ you found above, prove that
the mapping $T : V \rightarrow \mathbb{R}^n$ given by $T(v) = [v]_B$ is linear.
3. (13 pts) Let $T : V \to W$ be linear where $V$ and $W$ are vector spaces.

(a) Define the set $\ker T$, the kernel of $T$.

(b) $\ker T$ is a subset of the vector space ______.

(c) Prove that $\ker T$ is a subspace of the vector space you named above.
4. (9 pts) Let $A$ be a $2 \times 2$ matrix. Let $T$ be the map $x \mapsto Ax$, i.e. the map $T(x) = Ax$. Plot and label the following vectors in the given plane.

(a) $T(x)$ where $x = (4, 2)$ is an eigenvector of $A$ with eigenvalue $\lambda = -2$.

(b) An eigenvector $u \neq x$ in the same eigenspace as $x$ above.

(c) $T(y)$ where $y = (-2, 3)$ is an eigenvector of $A$ with eigenvalue $\lambda = 3$. 
5. (8 pts) Let \( A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 3 & 2 \\ -2 & 0 & 1 \end{pmatrix} \).

Find a basis for the eigenspace of \( A \) corresponding to the eigenvalue \( \lambda = 3 \).

This is equivalent to finding a basis for \( \text{Nul} (\quad) \). (fill in the blank.)
6. (10 pts) Let \( S = \{ p_1 = 1 + t^2, \ p_2 = t + t^2, \ p_3 = 2 - 3t - t^2, \ p_4 = 4 + t + 5t^2 \} \) be a set of polynomials in \( \mathbb{P}_2 \).

(a) The dimension of \( \mathbb{P}_2 \) is ____ so \( \mathbb{P}_2 \) is isomorphic to \( \mathbb{R}____ \). (fill in the blanks)

(b) Do the polynomials in \( S \) span \( \mathbb{P}_2 \)? Justify completely by explaining your method.
7. (10 pts) Let \( B = \{ b_1, b_2 \} \) where \( b_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \) and \( b_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \).

(a) Find \( P_B \), the change-of-coordinates matrix from \( B \) to the standard basis in \( \mathbb{R}^2 \).

(b) Suppose \( x = \begin{bmatrix} 8 \\ 2 \end{bmatrix} \). Find the coordinates of \( x \) relative to \( B \), i.e. find \( [x]_B \).
8. (10 pts) Let $A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 \end{pmatrix}$.

(a) Find $\det A$.

(b) Find $\det(2A^3)$. 
9. (10 pts) Let $T : \mathbb{R}^4 \rightarrow M_{2 \times 3}$ be defined by

$$T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} 3a & a + c & 0 \\ 0 & 0 & 2c \end{bmatrix}.$$ 

Then $T$ is linear. (You do not need to show this.)

(a) Find a basis for the range of $T$. Show work but no justification of basis properties is needed.

(b) Find a basis for ker $T$. Show work but no justification of basis properties is needed.

(c) Is $T$ one-to-one? Briefly justify your answer.
10. (8 pts) A linear transformation is called onto if for every vector \( w \in W \) there is at least one vector \( v \in V \) with \( T(v) = w \). Suppose that the linear transformation \( T : V \to W \) is onto. Let \( S = \{v_1, v_2, v_3\} \) be a spanning set for \( V \). Show that \( S' = \{T(v_1), T(v_2), T(v_3)\} \) is a spanning set for \( W \). The first line of your proof should be: "Let \( w \) be any vector in \( W \)."
Name ________________________________

Please circle your section

Section 1    Hao Liang

Section 2    Mary Glaser

I pledge that I have neither given nor received assistance on this exam.

Signature ________________________________