1. (10 points) True/False. Please write the word "true" or "false".

   (i) If $A$ and $B$ are two $n \times n$ matrices such that $\det(AB) = 4$, then $A$ and $B$ are both invertible.

   (ii) If $H$ is a subset of a vector space $V$ such that $\bar{0} \in H$, then $H$ is a subspace of $V$.

   (iii) If $A$ is an $n \times n$ matrix and $v \in \mathbb{R}^n$ is a nonzero vector, then the set of solutions of the matrix equation $Ax = v$ is a subspace of $\mathbb{R}^n$.

   (iv) If a set of vectors $\{v_1, \ldots, v_m\}$ in a vector space $V$ spans $V$, then $\dim V > m$.

   (v) The vector space $\mathbb{P}$ of all polynomials in one variable is infinite-dimensional.

2. (10 points) Set $B = \{v_1, v_2\}$, where $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   (i) Using only determinants, show that $B$ is a basis of $\mathbb{R}^2$ (do not use row operations).

   (ii) If $[v]_B = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, find $v$.

   (iii) Find the change-of-coordinates matrix $P_B$ (recall that $P_B$ is the matrix such that $v = P_B[v]_B$, for all $v \in \mathbb{R}^2$).
3. (10 points) Let $V$ be a finite-dimensional vector space with basis $B = \{v_1, \ldots, v_n\}$.

(i) If $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, then what is $[v]_B$?

(ii) The coordinate mapping $v \mapsto [v]_B$ is a linear map from $V$ to what vector space?

(iii) Show that the coordinate mapping $v \mapsto [v]_B$ is one-to-one by choosing two arbitrary vectors $u$ and $v$ and showing that if $[u]_B = [v]_B$ then $u = v$. 

4. (6 points) Determine whether each of the following is a subspace of $P_2$. You do not need to show your work; no work will be graded.

(i) $\{a + bt + ct^2 : b = 0\}$.

(ii) $\{a + bt + ct^2 : b = c\}$.

(iii) $\{a + bt + ct^2 : a = 3\}$.

5. (14 points) Determinants.

(i) Find the determinant of $A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 0 & 1 \\ 9 & 6 & 1 \end{pmatrix}$ using cofactor expansion.

(ii) Suppose that $A$ is a square matrix with $\det A = 4$. Find the determinant of $B$, where the matrix $B$ is obtained from $A$ by each of the following operations:

(a) Switching two rows of $A$.

(b) Multiplying a row of $A$ by $-\frac{1}{3}$.

(c) Adding twice the first row of $A$ to the third row of $A$.

(d) Multiplying $A$ by itself 3 times (i.e., $B = A^3$).

(e) Inverting $A$. 

3
6. (12 points) Linear independence.

(i) Define what it means for a collection of vectors \( \{v_1, \ldots, v_m\} \) in a vector space \( V \) to be linearly independent.

(ii) Let \( B = \{v_1, \ldots, v_n\} \) be a basis of a vector space \( V \) and let \( v \in V \) be any vector. Show that the set \( \{v_1, \ldots, v_n, v\} \) is linearly dependent by finding a dependence relation.

(iii) Let \( B = \{1, t, t^2\} \) be the standard basis of \( \mathbb{P}_2 \). Use the coordinate mapping associated to \( B \) to determine whether or not the set

\[
S = \{1 + 2t + 3t^2, 4 + 5t + 6t^2, 2 + t\}
\]

is linearly independent.
7. (12 points) Let \( A = \begin{pmatrix} -3 & -2 & 12 & -4 \\ 7 & 7 & -12 & 11 \\ -4 & -7 & -11 & -9 \\ -1 & 0 & -1 & -7 & -1 \end{pmatrix} \). Then \( A \) is row equivalent to the matrix

\[
B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

(you do not have to verify this). Find bases for

(i) \( \text{row } A \)

(ii) \( \text{col } A \)

(iii) \( \text{nul } A \)

(iv) \( \text{nul } B \)
8. (14 points) Suppose that $A$ is a $4 \times 11$ matrix. Find the following:

(i) The maximum value of rank $A$.

(ii) The minimum value of rank $A$.

(iii) The maximum value of $\dim(\text{nul } A)$.

(iv) The minimum value of $\dim(\text{nul } A)$.

(v) Row $A$ is a subspace of ________.

(vi) Col $A$ is a subspace of ________.

(vii) Nul $A$ is a subspace of ________.

9. (12 points) Let $V$ and $W$ be vector spaces, and let $T: V \rightarrow W$ be a linear map.

(i) Define the kernel of $T$.

(ii) Define the range of $T$.

(iii) Prove that the kernel of $T$ is a subspace of $V$. **WARNING:** Do NOT be tempted to use matrices.