2) \( \frac{dx}{dt} - \frac{1}{2t} x = \frac{1}{2} \quad t \to 0 \quad (N) \)

(a) \( \frac{dx}{dt} - \frac{1}{2t} x = 0 \) (H)

(H) solved using SOV: \( \frac{dx}{x} = \frac{1}{2t} \, dt \)

integrating...

\[ \ln|x| = \frac{1}{2} \ln t + c \quad \Theta \frac{1}{2} \ln(t^2) + c \]

\[ \Rightarrow x = c \sqrt{t} \]

Notice: \( x=0 \) can be achieved \( \lim_{t \to 0} x = 0 \)

Try guess \( x(t) = k(t)^{\frac{1}{2}} \)

\[ x'(t) = k'(t)^{\frac{1}{2}} + \frac{1}{2} k(t)^{-\frac{1}{2}} k(t) \]

Then \( x' - \frac{1}{2t} x = k'(t)^{\frac{1}{2}} + \frac{1}{2} k(t)^{-\frac{1}{2}} k(t) - \frac{1}{2} k(t)^{\frac{1}{2}} \)

Using (N) \( \Theta \frac{1}{2} \)

So \( k'(t) = \frac{1}{2t^2} \)

Thus \( k(t) = \frac{1}{t} \)

So particular soln is \( \sqrt{t^2} \)

Geol soln: \( x(t) = x_k(t) + x_p(t) \)

\[ = c \sqrt{t} + t \]

(b) Given \( x(1) = 0 \). Notice \( x(1) = c + 1 \)

So \( c = -1 \), and specific solution is

\[ x(t) = t - \sqrt{t^2} \]
3) \( \frac{dx}{dt} = x + x^{1/3} \) \( \quad x(0) = 0 \)

(a) Let \( F(t,x) = x + x^{1/3} \)

Then \( f_x(t,x) = 1 + \frac{1}{3x^{2/3}} \quad \Rightarrow \quad f_x(t,x) \) is not continuous at \((0,0)\)

Since \( f_x(t,x) \) is not continuous at the initial condition, a unique solution is not guaranteed by the EUT.

(b) To solve, use SOV \( \frac{dx}{x + x^{1/3}} = dt \) \( \quad \text{assuming } x \neq 0 \)

Let \( u = x^{1/3} \)

\( \Rightarrow \quad x = u^3 \)

\( \Rightarrow \quad dx = 3u^2 \, du \)

so \( \int \frac{dx}{x + x^{1/3}} = \int \frac{3u^2 \, du}{u^3 + u} = \int \frac{3u}{u^2 + 1} \, du \)

Let \( w = u^2 + 1 \)

\( dw = 2udu \)

\( \int \frac{dw}{w} \)

\( \Rightarrow \quad \frac{3}{2} \int \frac{dw}{w} \)

\( = \frac{3}{2} \ln |w| \quad \text{always positive} \)

\( = \frac{3}{2} \ln (u^2 + 1) \)

\( = \frac{3}{2} \ln (x^{2/3} + 1) \)

So \( \frac{3}{2} \ln (x^{2/3} + 1) = t + C \)

\( \ln (x^{2/3} + 1) = \frac{2}{3}t + C \)

\( x^{2/3} + 1 = ce^{2t/3} \)

\( x(t) = \left( ce^{2t/3} - 1 \right)^{3/2} \quad \text{Notice } x = 0 \text{ cannot be achieved for any value of } c \)

\( x(0) = (c - 1)^{3/2} \quad \text{IC } \quad \Rightarrow \quad 0 \quad \text{ so } c = 1 \)

Specific solutions are \( x(t) = (e^{2t/3} - 1)^{3/2} \) and \( x(t) = 0 \) since
(a) 

\[(e^2 - 2 + 2)D^3 - e^2 D^2 + 2 + 2 = 0\] 

\(-\infty < \gamma < \infty\)

(b) Yes (b/c all coeff func are continuous for all x in \(\mathbb{R}\))

and \(e^2 - 2t + 2\) is never zero (the discriminant \((b^2 - 4ac)\) is negative)

(c) 

\[h_0(t) = e^t\]

\[h'_0 = e^t\]

\[h''_0 = 3e^t\]

\[h'''_0 = 6e^t\]

\[L = (e^2 - 2 + 2)D^3 - e^2 D^2 + 2 + 2 = 0\]

\[Lh_0 = 0 - t^2(2) + 2t(2t) - 2(e^t) = -2t^2 + 4t^2 - 2e^t\]

\[= 0\]

So \(h_0\) is a soln.

\[h_2(t) = e^t\]

\[h'_2 = e^t\]

\[h''_2 = 2e^t\]

\[h'''_2 = 3e^t\]

\[Lh_2 = (e^2 - 2 + 2)e^t - e^2 e^t + 2t e^t - 2e^t\]

\[= 0\]

So \(h_2\) is a soln.

\[h_3(t) = e^t\]

\[h'_3 = e^t\]

\[h''_3 = e^t\]

\[h'''_3 = e^t\]

\[Lh_3 = (e^2 - 2 + 2)e^t - e^2 e^t + 2t e^t - 2e^t\]

\[= 0\]

So \(h_3\) is a soln

(c) \(W[h_0, h_2, h_3](t) = \det\begin{bmatrix}e^t & e^t & e^t \\ e^t & e^t & e^t \\ e^t & e^t & e^t \end{bmatrix}\)

\[W[h_0, h_2, h_3](0) = \det\begin{bmatrix}0 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} = (-1) \det\begin{bmatrix}1 & 0 \\ 0 & 2 \end{bmatrix} = -2\]

Since \(W[h_0, h_2, h_3](t_0) \neq 0\) for a \(t_0\) in \(-\infty, \infty\),

Wronskian test asserts \(h_0, h_2, h_3\) are linearly independent.
(d) \( c_1 h_1 + c_2 h_2 + c_3 h_3 = 0 \)

\[ \Rightarrow c_1 e^t + c_2 e^{-t} + c_3 e^t = 0 \quad (\star) \]

choose \( t = -1 \)

\[ \Rightarrow -c_1 + c_2 + c_3 e = 0 \quad (1) \]

\( t = 0 \)

\[ \Rightarrow c_3 = 0 \quad (2) \]

\( t = 1 \)

\[ \Rightarrow c_1 + c_2 + c_3 e = 0 \quad (3) \]

Eq'n (2) asserts \( c_3 = 0 \) substituting \( c_3 = 0 \) in eqns (1) and (2) yields

\[ -c_1 + c_2 = 0 \quad (4) \]

\[ c_1 + c_2 = 0 \quad (5) \]

Adding yields \( 2c_2 = 0 \) so \( c_2 = 0 \).

Substituting \( c_2 = c_3 = 0 \) in eqn (3) yields \( c_1 = 0 \).

So \( c_1 = c_2 = c_3 = 0 \) is only sol'n to eqns (1), (2), and (3); necessary for \( E(t) \) to be true.

So \( h_1, h_2, h_3 \) are linearly独立.

(e) Since \( h_1, h_2, h_3 \) are all sol'n's to \( Lx = 0 \)

and \( h_1, h_2, h_3 \) are linearly independent

and the order of \( Lx = 0 \) is 3 (same # of linearly indep sol'n's)

the linear combination \( x(t) = c_1 h_1(t) + c_2 h_2(t) + c_3 h_3(t) \)

\[ = c_1 e^t + c_2 e^{-t} + c_3 e^t \]

generates the general sol'n to \( Lx = 0 \).