Compensation Handout

Three math students (named A, B and C) share a book, and when class is over, have to decide who gets it. We'll look at 5 different possible compensation arrangements, and then compare them.

Suppose these are the values each student has for the book:

\[
\begin{align*}
    a &= \$18 \\
    b &= \$24 \\
    c &= \$30
\end{align*}
\]

1) Suppose C gets the book and pays A and B each $2. What is C's payout?

\[
\chi_c = c - \chi_a - \chi_b = 26
\]

 envy-free? not fair => not envy-free

fair? \( \text{No} \) \( 2 \times \frac{18}{3} = 6 \)

not fair to A (or B)

equitable? \( \text{No} \) \( \frac{c}{\omega} = \frac{30}{5} > \frac{m}{3} \)

Prop 14.9: equitable arrangement must be fair. Since this arrangement is not fair, it cannot be equitable (in this case).

Pareto-optimal? \( \text{Yes} \) C is a highest bidder (Prop 14.2)

2) Suppose C gets the book and pays A and B each $8. What is C's payout?

\[
\begin{align*}
    \chi_c &= 30 - 16 = 14 \\
    \chi_a &= 8 \\
    \chi_b &= 8
\end{align*}
\]

envy-free? \( \text{Yes} \)

fair? \( \text{Yes} \)

\[
\frac{\chi_c}{c} = \frac{14}{30} > \frac{\chi_a}{a} = \frac{8}{18}
\]

\[
\frac{\chi_c}{c} = \frac{14}{30} = \frac{7}{15} > \frac{\chi_a}{a} = \frac{8}{18}
\]

Prop 14.2: \( \text{Yes} \) again by prop 14.2

Pareto-optimal?

3) For an equitable and fair arrangement, who can get the book? B and C are both average so we could give it to either and find an equitable arrangement.

Suppose B gets the book. Find an equitable, and fair arrangement.

\[
\begin{align*}
    q &= \frac{w}{s} = \frac{24}{18 + 24 + 30} = \frac{24}{72} = \frac{1}{3} \\
    q_a &= \frac{q \cdot a}{3} = \frac{1}{3} \cdot 18 = 6 \\
    q_b &= \frac{q \cdot b}{3} = \frac{1}{3} \cdot 24 = 8 \\
    q_c &= \frac{q \cdot c}{3} = \frac{1}{3} \cdot 30 = 10
\end{align*}
\]

is this envy-free? \( \text{No} \)

Pareto-optimal? \( \text{No} \) B is not a highest bidder (Prop 14.2)
4) Find a Pareto-optimal, fair, equitable arrangement. Who gets the book?

\[
\begin{align*}
\text{paychecks: } & \sum X_A = \frac{5}{12} \times 18 = 15 \frac{1}{2} \\
& X_B = \frac{5}{12} \times 24 = 10 \\
\text{value of book to } C - X_C = & \frac{5}{12} \times 30 = 25 \frac{1}{2} \\
\text{check: } & X_A + X_B + X_C = 60 \\
\text{is this envy free? } & \text{No, } A \text{ and } B \text{ get different paychecks.}
\end{align*}
\]

is this an objective improvement over arrangement 3?

\text{Yes, literally everyone got a bigger payout.}

5) Find all Pareto-optimal, fair, and envy-free arrangements:

\[
\begin{align*}
& X_B = \frac{2}{3} \times 8 \\
& X_C = \frac{2}{3} \times 6 \\
& X_A = \frac{2}{3} \times 10
\end{align*}
\]

Pareto-optimal requires...

envy-free requires...

\[
X_A = X_B \text{ and } \frac{b}{3} \leq X_B \leq \frac{c}{3} = 10
\]

Prop 13.10: all compensation amounts equal and the arrangement is fair.

these are described by

\[
X_B = X_A, \quad 8 \leq X_B \leq 10, \quad X_C = 30 - X_A - X_B
\]

6) Looking at all the arrangements, there are two categories of better arrangements:

What arrangements were fair, Pareto-optimal, and equitable?

\# 4

What arrangements were fair, Pareto-optimal, and envy-free?

\# 2, everything in \# 5 (note that the payout in \# 2 represents the intersection of \( X_A = X_B \) with \( X_B = 8 \))

Is a fair, Pareto-optimal, equitable, envy-free arrangement possible for this book?

\text{Nope. the fair, Pareto-optimal, equitable arrangement in 4 is uniquely determined, meaning there is only one. Since it was not envy-free, we can never have all the things (tragedy).}