Solving linear equations.

One method to put a matrix A into reduced row echelon form (RRE):

NOTE: The *leading entry* of a row of a matrix is the left-most non-zero entry of that row.

1. Be clever. Then, put all rows consisting entirely of zeros at the bottom of the matrix.

2. Choose a simple row with a leading entry in the first column. Switch this row with the first row. Use this row and elementary row operations to get all entries in the first column below the first row equal to zero. (I say *clean down* the first column).

3. Now, forget about this first row. Choose a simple row with a leading entry in the second column. Switch this row with the second row. Use this row and elementary row operations to make all entries below its leading entry equal to zero (clean down the second column).

4. Keep cleaning down successive columns.

The matrix is now in row echelon form (but not necessary reduced row echelon form). The leading entries are in the pivot positions for A.

5. *Clean up.* That is, use the lowest non-zero row and elementary row operators to make all entries in the column above its leading entry zero. Make that leading entry equal to one using a row operation.

6. Now, forget about this lowest row and use the row above it to clean up, making all entries in the column above its leading entry zero. Then, make its leading entry equal to one using a row operation.

7. Continue cleaning up. Then, make sure all leading entries are equal to one and the matrix is in RRE.

To solve $m \times n$ linear algebraic equations $A\boldsymbol{x} = \boldsymbol{b}$:

1. Form the augmented matrix $[A\,\boldsymbol{b}]$ and reduce to RRE.

2a. If in this process you get a leading entry in the last column (the matrix has a row: $(0, \ldots, 0, b)$ where $b \neq 0$) then the system is inconsistent. (That is, there are no solutions to the equations because one of the equations is $0 = b$.)

2b. If not, continue to reduce to RRE, convert to equations and solve for the *leading variables (or basic variables)* (variables corresponding to leading entries of the reduced matrix) in terms of the *free variables* (non-leading variables) and constants. Note that the leading (basic) variables correspond to pivot columns of A.

To solve the $m \times n$ homogeneous linear algebraic equations $A\boldsymbol{x} = \mathbf{0}$:

1. Reduce the augmented matrix $[A\,\mathbf{0}]$ to RRE. (You can reduce A as long as you convert back to **homogeneous** equations.)

2. Solve for the basic (leading) variables in terms of free variables (if there are free variables).

3. If there are free variables: successively set each free variable equal to one (or an 'easy' non-zero number) and set the other free variables equal to zero. Then, write your solutions as vectors. This gives a list of **linearly independent** solution vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$(one for each free variable). Every solution to $A\boldsymbol{x} = \mathbf{0}$ is a linear combination of these solution vectors; therefore, the solution set to $A\boldsymbol{x} = \mathbf{0}$ is $\text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$.