Dec 14, 2000

SOLUTIONS (Provided by students in Math 46 - 01, Spring 2012)

1. (a) As A is lower triangular it is easy to compute its eigenvalues: these are just the entries on its main diagonal, i.e $\lambda = 1, 2, 0$.

   (b) For $\lambda = 1$,

$$A - I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   We therefore have a one parameter family of solutions for this system, where x_1 is free, and $x_2 = x_3 = 0$. An eigenvector associated to the eigenvalue 1 is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

   For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   We therefore have a one parameter family of solutions for this system, where $x_1 = -x_2$, x_2 is free, and $x_3 = 0$. An eigenvector associated to the eigenvalue 2 is $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

   For $\lambda = 0$,

$$A - 0I = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   We therefore have a one parameter family of solutions for this system, where $x_1 = x_2$ and $2x_2 = -x_3$. Then, $x_1 = -\frac{1}{2}x_3$, $x_2 = -\frac{1}{2}x_3$, and x_3 is free. An eigenvector associated to the eigenvalue 0 is

$$\vec{v}_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

   (c) The matrix A is diagonalizable because the eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ associated to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$, respectively, are linearly independent.

2. As $[p(t)]_S$ is a polynomial with coordinate vector $[p(t)]_S = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$, it can be written as

$$\begin{aligned} p(t) &= 2 \cdot [1 - t] - 1 \cdot [2t + t^2] + 5 \cdot [3t^2] \\ &= 14t^2 - 4t + 2. \end{aligned}$$

Note that in this question, it is enough to write $p(t)$ explicitly, what corresponds to writing the polynomial in terms of the canonical basis $\{1, t, t^2\}$.

3. (a) To compute the determinant, we can expand down the second column

$$\begin{vmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 2 & -2 & 1 \\ 2 & 0 & 5 & 3 \end{vmatrix} = -0 \cdot \begin{vmatrix} 1 & 3 & 1 \\ 0 & -2 & 1 \\ 2 & 5 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 2 & 5 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 2 & 5 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & -2 & 1 \end{vmatrix}$$
$$= -11$$

(b) As the determinant of A is different from 0, the matrix is invertible.

4. (a) $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$. The matrix

$$[L]_{\mathcal{S} \leftarrow \mathcal{S}} = \left[[L(\mathbf{s}_1)]_{\mathcal{S}} \quad [L(\mathbf{s}_2)]_{\mathcal{S}} \right].$$

We therefore compute the images of the vectors of the basis $\mathcal{S}$, and then their expression in the basis $\mathcal{S}$:

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\mathbf{s}_1 + 3\mathbf{s}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathcal{S}}$$
$$L\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1\mathbf{s}_1 - 3\mathbf{s}_2 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}_{\mathcal{S}}$$

Therefore, the matrix

$$[L]_{\mathcal{S} \leftarrow \mathcal{S}} = \begin{bmatrix} 2 & -1 \\ 3 & -3 \end{bmatrix}.$$

(b) The matrix P is just the change-of-basis matrix $P_{\mathcal{S} \leftarrow \mathcal{T}}$. To compute that, we just express the elements in the basis $\mathcal{T}$ as vectors in the basis $\mathcal{S}$:

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}_{\mathcal{S}}$$
$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}_{\mathcal{S}}$$

The matrix is therefore

$$P_{\mathcal{S} \leftarrow \mathcal{T}} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}.$$

We can check our work just by computing the inverse of $P_{\mathcal{S} \leftarrow \mathcal{T}}$ and then the product $P^{-1}[L]_{\mathcal{S} \leftarrow \mathcal{S}} P$. Using the algorithm to find the inverse of a matrix, we obtain

$$P^{-1}_{\mathcal{S} \leftarrow \mathcal{T}} = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}.$$

The product is $\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}$, what is in fact the matrix $L_{\mathcal{T} \leftarrow \mathcal{T}}$.

5. (a) If we want a vector that is in the kernel of the linear transformation L, we just need to find out which vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ are mapped to $\vec{0}$ by L.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 3a - b \\ -3a + b \end{bmatrix} \Rightarrow \begin{bmatrix} 3a - b \\ -3a + b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3a = b.$$

A vector satisfying this condition, and hence in the kernel of the linear transformation L would be $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. It can be easily checked that the image of this particular vector under L is the zero vector.

(b) The standard matrix of this linear transformation is $A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$. Let us try to solve the system $A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by row-reduction

$$\begin{bmatrix} 3 & -1 & 1 \\ -3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Since $0 \neq 3$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not in the range of L.

6. (a) By row-reduction,

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 2 & 4 & 0 \\ 2 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -10 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -10 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we conclude that S is linearly dependent, since there are fewer pivots than columns.

(b) From the previous part, a basis for $\text{Span}(S)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -4 \\ 0 \end{bmatrix} \right\}.$$

(c) From part (b), $\dim \text{Span}(S) = 3$.

7. (a) False
   (b) True
   (c) False
   (d) False
   (e) True
   (f) False
   (g) False

(h) False

(i) False

(j) False

(k) True

8. Let A be the standard matrix for the linear transformation L. If 0 is an eigenvalue of L, i.e. an eigenvalue of its standard matrix, by the invertible matrix theorem we have that A is not invertible. And again, by the invertible matrix theorem, if a matrix A is not invertible, the linear transformation associated to it is not one-to-one.