Math 46
Final Exam
Spring 2006

Your name_____

Directions: For each problem, place the letter choice of your answer in the spaces provided on this page.

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By signing here, I pledge that I have neither given nor received assistance on this exam.

Your signature_____

1. Given that $A \sim B$, where

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the dimension of $\mathrm{Nul}(A)$ is

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

2. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \in V$ and $2\mathbf{v}_1 + 3\mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}$.

I. $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.
II. $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ are linearly dependent.
III. $\mathbf{v}_2, \mathbf{v}_5$ are linearly **in**dependent.

(a) Only I is true
(b) Only II is true
(c) I and II are true
(d) Only III is true
(e) I and III are true

3. Let $U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x + 2y = 0 \right\}$. Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y = 2x^2 \right\}$.

(a) Only U is a subspace of $\mathbb{R}^2$
(b) Only V is a subspace of $\mathbb{R}^2$
(c) Both U and V are subspaces of $\mathbb{R}^2$
(d) Neither U nor V is a subspace of $\mathbb{R}^2$

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^5$ be a linear transformation with $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then

(a) T is 1-1 but not onto
(b) T is onto but not 1-1
(c) T is 1-1 and onto
(d) T is neither 1-1 nor onto
(e) There is not enough information to decide

5. Suppose A is 10×12 and $T(\mathbf{x}) = A\mathbf{x}$. Suppose $\dim(\ker(T)) = 3$.

I T is not 1-1
II. T is not onto

(a) Only I is true
(b) Only II is true
(c) I and II are both true
(d) There's not enough information to decide

6. Let $A = \begin{bmatrix} 1 & a \\ a & 3a \end{bmatrix}$. Then A is **not** invertible if

   (a) $a = -3$ (b) $a = 0$ (c) $a = 0$ or $a = -3$ (d) $a = 0$ or $a = 3$

7. Determine the values of h and k so that

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix}$$

   is the augmented matrix of a consistent linear system with infinite solutions.

   (a) $h = 9, k = 6$
   (b) $h = 9, k \neq 6$
   (c) $h \neq 9, k \in \mathbb{R}$
   (d) $h \neq 9, k \neq 6$

8. Suppose V is a vector space with dimension n.

   I. Every set of $m < n$ vectors in V is linearly independent.
   II. Every set of $m > n$ vectors in V spans V.

   (a) Only I is true
   (b) Only II is true
   (c) I and II are both true
   (d) I and II are both false

9. Suppose $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ and $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = ||\mathbf{x}||$.

   I. $T : \mathbb{R}^2 \to \mathbb{R}^2$
   II. $T : \mathbb{R}^2 \to \mathbb{R}$
   III. T is a linear transformation
   IV. T is **not** a linear transformation

   (a) I and III are true
   (b) I and IV are true
   (c) II and III are true
   (d) II and IV are true

10. Let $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be a set of polynomials in $\mathbf{P}_3$, let A be the matrix whose columns are the coordinate vectors of the polynomials, and suppose $A \sim B$, where

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Then

   (a) S spans $\mathbf{P}_3$, but the polynomials in S are not linearly independent.
   (b) The polynomials in S are linearly independent, but S does not span $\mathbf{P}_3$
   (c) S is a basis for $\mathbf{P}_3$
   (d) S is an orthogonal basis for $\mathbf{P}_3$

11. **Warning:** Look at the elements of S very carefully.
Let

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + t \begin{bmatrix} 4 \\ -2 \\ 12 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ -18 \end{bmatrix} \right\}.$$

Then S is

(a) A plane in $\mathbb{R}^3$ not through $\mathbf{0}$
(b) A plane in $\mathbb{R}^2$ not through $\mathbf{0}$
(c) A line in $\mathbb{R}^3$ through $\mathbf{0}$
(d) A line in $\mathbb{R}^2$ not through $\mathbf{0}$

12. The dimension of the vector space $\mathbf{M}_{7\times 9}$ is

(a) 7 (b) 9 (c) 16 (d) 63

13. Let A and B be matrices. Suppose $T_1 : \mathbb{R}^3 \to \mathbb{R}^5$ is given by $T_1(\mathbf{x}) = A\mathbf{x}$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $T_2(\mathbf{x}) = B\mathbf{x}$. Then the map defined by $T(\mathbf{x}) = (AB)^T(\mathbf{x})$ is

(a) A linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^5$
(b) A function from $\mathbb{R}^5$ to $\mathbb{R}^2$ but not a linear transformation
(c) A linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^2$
(d) not defined

14. Let $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.
One eigenvector of A is

(a) $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$

15. For the eigenvalue $\lambda = 1$ of the matrix $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$, the eigenspace is k-dimensional, where $k =$

(a) 1 (b) 2 (c) 3 (d) 1 is not an eigenvalue of A

16. For an $n \times n$ diagonalizable matrix A, identify the statement that is *not* necessarily true.

(a) A has n linearly independent eigenvectors.

(b) The multiplicity of each eigenvalue of A is equal to the dimension of the corresponding eigenspace.

(c) $AP = PD$, where P is made of eigenvectors of A and D is a diagonal matrix whose entries correspond to the columns of P.

(d) The eigenvectors of A form a basis of $\mathbb{R}^n$.

(e) A has n distinct eigenvalues.

17. The matrix $A = \begin{bmatrix} -5 & -5 \\ 5 & -5 \end{bmatrix}$ has complex eigenvalues. For any $\mathbf{x} \in \mathbb{R}^2$, the computation $A\mathbf{x}$ serves to rotate $\mathbf{x}$ by an angle of $\frac{3\pi}{4}$ counterclockwise, and it scales $\mathbf{x}$ by a factor of...

(a) 5 (b) -5 (c) $5\sqrt{2}$ (d) The angle of rotation is *not* $\frac{3\pi}{4}$ counterclockwise.

18. A is a real 5×5 matrix with eigenvalues $\lambda = 1, 1, 2, 2,$ and k. The eigenspaces for $\lambda = 1$ and 2 are each two-dimensional. Then A is guaranteed to be diagonalizable if

(a) $k \neq 0$ (b) $k \neq 1,\ 2$ (c) $k \neq 0,\ 1,\ 2$ (d) k is complex

19. Let $\mathcal{B}$ be a basis for $\mathbb{R}^3$, let $\mathcal{D}$ be a basis for $\mathbf{P}_4$, and let $T : \mathbb{R}^3 \to \mathbf{P}_4$ be a linear transformation. Then the matrix of T relative to $\mathcal{B}$ and $\mathcal{D}$ is

(a) 5×3 (b) 4×3 (c) 3×5 (d) 3×4

20. Let $T : \mathbb{R}^4 \to \mathbf{M}_{2\times 2}$ be a linear transformation, and recall that the standard bases for $\mathbb{R}^4$ and $\mathbf{M}_{2\times 2}$ are $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, respectively.
Suppose the matrix of T relative to the standard bases of $\mathbb{R}^4$ and $\mathbf{M}_{2\times 2}$ is

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 0 & 5 & -2 \\ 3 & 0 & 9 & 3 \\ 1 & 4 & 0 & 1 \end{bmatrix}$$

Then

$$T\left(\begin{bmatrix} 5 \\ -1 \\ 3 \\ 12 \end{bmatrix} \right) =$$

(a) $\begin{bmatrix} -8 \\ 5 \\ 3 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -6 \\ 1 \\ 78 \\ 13 \end{bmatrix}$ (c) $\begin{bmatrix} -8 & 5 \\ 3 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -6 & 1 \\ 78 & 13 \end{bmatrix}$

21. Let $T : \mathbf{P}_3 \to \mathbf{P}_2$ be defined by $T(\mathbf{p}(t)) = \mathbf{p}'(t) + (t-3)\mathbf{p}''(t)$. Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis of $\mathbf{P}_3$ and let $\mathcal{D} = \{\ 1+t, 2-t, 1+t^2\}$ be a basis of $\mathbf{P}_2$. The third column of the matrix of T relative to $\mathcal{B}$ and $\mathcal{D}$ is

(a) $\begin{bmatrix} -6 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2/3 \\ -10/3 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} -6 \\ 4 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

22. Let $A = \begin{bmatrix} -3 & -3 \\ -5 & 2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$. A least squares solution to $A\mathbf{x} = \mathbf{b}$ is

(a) $\begin{bmatrix} 3 \\ -9 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$

23. Let $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$. Then $\text{proj}_{\text{Col}(A)}\mathbf{b} =$

(a) $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 40/7 \\ 4/7 \\ 36/7 \end{bmatrix}$ (d) $\begin{bmatrix} 8 \\ -8 \\ 16 \end{bmatrix}$

24. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \end{bmatrix}$, and suppose for some $\mathbf{b} \in \mathbb{R}^2$, $\text{proj}_{\text{Col}(A)}\mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.
The set of least squares solutions to $A\mathbf{x} = \mathbf{b}$ is a

(a) point (b) plane *not* through the origin (c) plane through the origin (d) line *not* through the origin

25. Let A be any $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$.

I. There will always be at least one least squares solution to $A\mathbf{x} = \mathbf{b}$, even if $\mathbf{b}$ is not in $\text{Col}(A)$.
II. A least squares solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$ is the closest point to $\mathbf{b}$ in $\text{Col}(A)$.
III. If $\hat{\mathbf{x}}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$, then $||A\hat{\mathbf{x}} - \mathbf{b}||$ is the least squares error.

(a) Only II is true
(b) Only III is true
(c) II and III are true
(d) I and III are true
(e) I, II, and III are all true