Math 46

Tufts University
Department of Mathematics
Final Exam

December 14, 2000

1. (15 points) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   (a) Find the eigenvalues of A.

   (b) For each eigenvalue, find an associated eigenvector.

   (c) Is A diagonalizable? Why or why not?

2. (5 points) Let $V = P_2$ and let S be the ordered basis

$$S = \{1 - t, 2t + t^2, 3t^2\}.$$

   Given that $p(t)$ is a polynomials with coordinate vector

$$[p(t)]_S = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix},$$

   find $p(t)$.

3. (8 points)

   (a) Find the determinant

$$\det \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 2 & -2 & 1 \\ 2 & 0 & 5 & 3 \end{bmatrix}$$

   (b) Is A invertible? Why or why not?

4. (15 points) Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a + b \\ a - 2b \end{bmatrix}.$$

   Let $S = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $\mathcal{T} = \left\{\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$.

   (a) Find $[L]_{S \leftarrow S}$, the matrix of L with respect to S.

   (b) Find a matrix P so that $[L]_{\mathcal{T} \leftarrow \mathcal{T}} = P^{-1}[L]_{S \leftarrow S}P$.

5. (10 points) Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 3a - b \\ -3a + b \end{bmatrix}.$$

   (a) Find a nonzero vector in Ker$L$.

   (b) Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the range of L? Why or why not?

6. (15 points) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (a) Prove or disprove that D is linearly independent.

   (b) Find a basis for Span(S).

   (c) Find the dimension of Span(S).

7. (22 points) Answer true or false (no explanations needed)

   (a) The set of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $3ab = 0$ is a subspace of $\mathbb{R}^3$.

   (b) The set of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $3a + b = 0$ is a subspace of $\mathbb{R}^3$.

   (c) The set of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $a + b + 3 = 0$ is a subspace of $\mathbb{R}^3$.

   (d) The map $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} ab \\ b \end{bmatrix}$ is linear.

   (e) The map $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a + b \\ b \end{bmatrix}$ is linear.

   (f) The map $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} \sqrt[3]{a} \\ b \end{bmatrix}$ is linear.

   (g) If $W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, then $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

   (h) For any $n \times n$ matrix A and vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, if $A\vec{x} = A\vec{y}$, then $\vec{x} = \vec{y}$.

   (i) If $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent subset of $\mathbb{R}^5$, and $\{\vec{v}_3, \vec{v}_4\}$ is a linearly independent subset of $\mathbb{R}^5$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent.

   (j) Let $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a linear transformation. If the dimension of the kernel of L is 1, then L maps $\mathbb{R}^4$ onto $\mathbb{R}^3$.

   (k) The intersection of two subspaces of a vector space cannot be empty

8. (10 points) Let $L : V \to V$ be a linear transformation. Given that 0 is an eigenvalue of L, prove that L is **not** one-to-one.