KEY

Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics
Exam II

November 16, 2015
Sections 1 and 2

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. **Please reference $\vec{0}$ with the vector space to which it belongs, e.g. the zero vector in V should be subscripted O_V.** *You are required to sign your exam.With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	12	
3	13	
4	9	
5	8	
6	10	
7	10	
8	10	
9	10	
10	8	
	100	

1. (10 points)

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then A and B are row equivalent. (You do not need to show this.)

(a) Find a basis $\mathcal{B}$ for Col A.

Use pivot cols of A which are determined from B (RREF of A)

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(b) Show that Col $A \neq$ Col B by finding a vector v that is in Col A but not Col B. Briefly justify.

The first column of A, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, is not in Col B because

the last 2 entries of every vector in Col B is 0.

(c) Find the dimension of Nul A.

# of columns of A = dim col A + dim Nul A

$5 = 3 + \boxed{2}$

So dim Nul A = 2

2. (12 pts) For this whole problem let V be a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

(a) V is isomorphic to $\mathbb{R}^n$ where $n =$ 2.

(b) Using the n you found above, prove that the mapping $T : V \to \mathbb{R}^n$ given by $T(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ is linear.

Let $u_1, u_2 \in V$. Then $\exists$ unique scalars c_1, c_2, d_1, d_2 such that $u_1 = c_1 b_1 + c_2 b_2$ and $u_2 = d_1 b_1 + d_2 b_2$ since $\mathcal{B}$ is a basis for V.

(1) Show $[\ ]_{\mathcal{B}}$ preserves $+$ (We must show $[u_1 + u_2]_{\mathcal{B}} = [u_1]_{\mathcal{B}} + [u_2]_{\mathcal{B}}$)

Given u_1, u_2 above,

$$[u_1 + u_2]_{\mathcal{B}} = [(c_1 u_1 + c_2 u_2) + (d_1 u_1 + d_2 u_2)]_{\mathcal{B}}$$

$$= [(c_1 + d_1) u_1 + (c_2 + d_2) u_2]_{\mathcal{B}}$$

$$= \begin{pmatrix} c_1 + d_1 \\ c_2 + d_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = [u_1]_{\mathcal{B}} + [u_2]_{\mathcal{B}}$$

(2) Show $[\ ]_{\mathcal{B}}$ preserves scalar multiplication.

Let $u \in V$. Then $\exists$ unique c_1, c_2 with $u = c_1 b_1 + c_2 b_2$.
Let k be a constant
We must show $[ku]_{\mathcal{B}} = k[u]_{\mathcal{B}}$.

$$[ku]_{\mathcal{B}} = [k(c_1 b_1 + c_2 b_2)]_{\mathcal{B}} = [(kc_1) b_1 + (kc_2) b_2]_{\mathcal{B}} = \begin{pmatrix} kc_1 \\ kc_2 \end{pmatrix} = k \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= k[u]_{\mathcal{B}}$$

$\therefore [\ ]_{\mathcal{B}} : V \to \mathbb{R}^2$ is linear

3. (13 pts) Let $T : V \to W$ be linear where V and W are vector spaces.

(a) Define the set ker T, the kernel of T.

$\ker T = \{ v \in V \mid T(v) = 0_W \}$

(b) Ker T is a subset of the vector space V.

(c) Prove that ker T is a subspace of the vector space you named above.

(1) $0_V \in \ker T$ because since T is linear $T(0_V) = 0_W$.

(2) Let $u, v \in \ker T$ We must show $u+v \in \ker T$ ($\Leftrightarrow T(u+v) = 0_W$)

$T(u+v) = T(u) + T(v) = 0_W + 0_W = 0_W$ $\therefore u+v \in \ker T$

$\uparrow$ T is linear $\quad \uparrow$ $u, v \in \ker T$

(3) Let $u \in \ker T$ and c be a scalar. We must show $cu \in \ker T$ ($\Leftrightarrow T(u) = 0_W$)

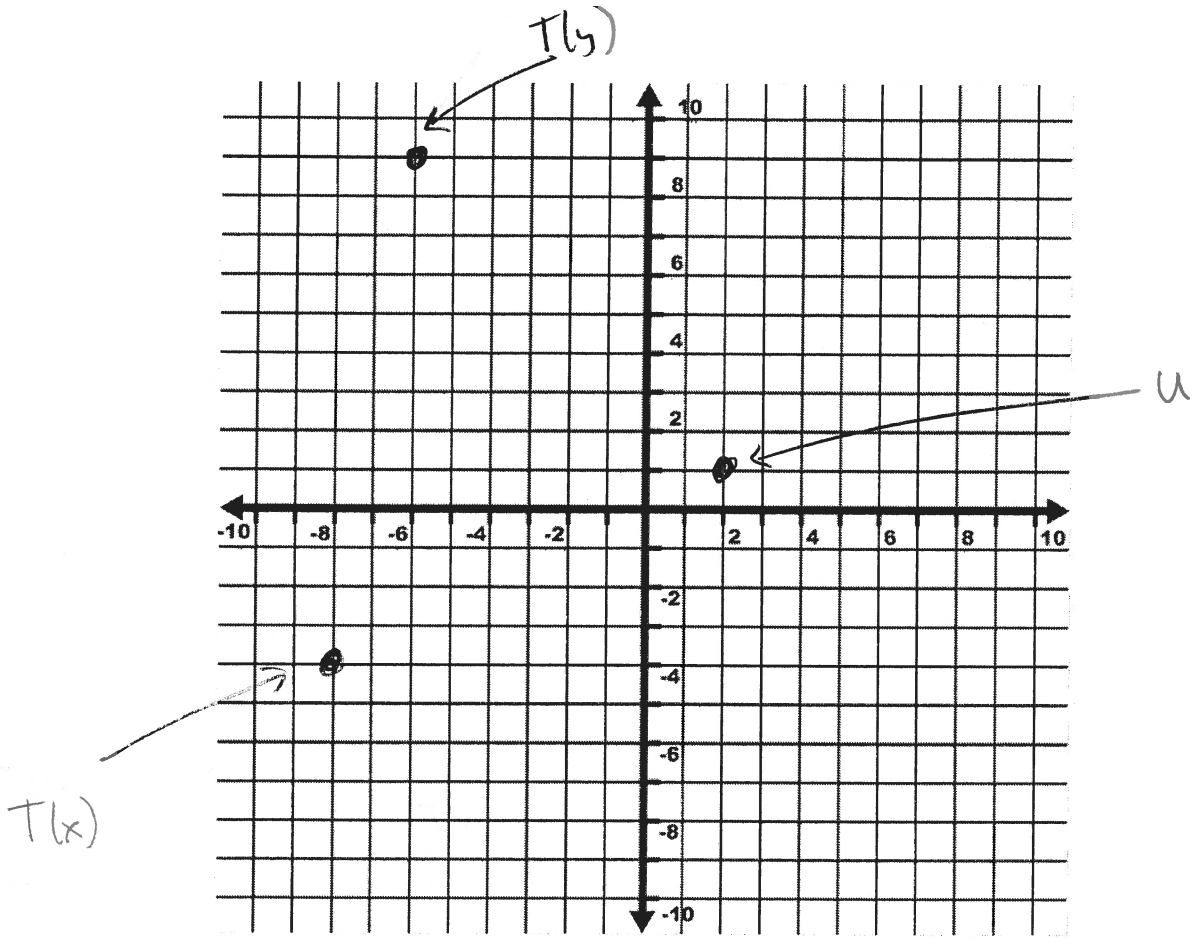
$T(cu) = c\,T(u) = c \cdot 0_W = 0_W$.

$\uparrow$ T is linear $\quad \uparrow$ $u \in \ker T$

$\therefore cu \in \ker T$

We have shown that $\ker T$ is a subspace of V. $\boxtimes$

4. (9 pts) Let A be a 2×2 matrix. Let T be the map $x \mapsto Ax$, i.e. the map $T(x) = Ax$. Plot and *label* the following vectors in the given plane.



(a) $T(x)$ where $x = (4, 2)$ is an eigenvector of A with eigenvalue $\lambda = -2$.

$T(x) = -2(4,2) = (-8,-4)$

(b) An eigenvector $u \neq x$ in the same eigenspace as x above.

Use any non-0 multiple of x, e.g. $\frac{1}{2}x = (2,1)$

(c) $T(y)$ where $y = (-2, 3)$ is an eigenvector of A with eigenvalue $\lambda = 3$.

$T(y) = 3(-2,3) = (-6,9)$

5. (8 pts) Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 3 & 2 \\ -2 & 0 & 1 \end{pmatrix}$.

Find a basis for the eigenspace of A corresponding to the eigenvalue $\lambda = 3$.

This is equivalent to finding a basis for Nul ($A - 3I$). (fill in the blank.)

Solve $(A - 3I)\vec{x} = \vec{0}$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right) \sim \begin{array}{c} \begin{array}{ccc} & f & f \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$$\begin{array}{c|c} x_1 + x_3 = 0 & x_1 = -x_3 \\ x_2 = x_2 & x_2 = x_2 \\ x_3 = x_3 & x_3 = x_3 \end{array}$$

$$\vec{x} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Basis: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

6. (10 pts) Let $S = \{p_1 = 1 + t^2,\ p_2 = t + t^2,\ p_3 = 2 - 3t - t^2,\ p_4 = 4 + t + 5t^2\}$ be a set of polynomials in $\mathbb{P}_2$.

(a) The dimension of $\mathbb{P}_2$ is $\underline{3}$ so $\mathbb{P}_2$ is isomorphic to $R^{\underline{3}}$. (fill in the blanks)

(b) Do the polynomials in S span $\mathbb{P}_2$? Justify completely by explaining your method.

Let $B = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$.

Then since $[\ ]_B : \mathbb{P}_2 \to \mathbb{R}^3$ is an isomorphism, the vectors in S span $\mathbb{P}_2$ if and only if $[p_1]_B, [p_2]_B, [p_3]_B, [p_4]_B$.

Let $A = \begin{bmatrix} [p_1]_B & [p_2]_B & [p_3]_B & [p_4]_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 1 \\ 1 & 1 & -1 & 5 \end{bmatrix} -R_1$

$\sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Since A does <u>not</u> have a pivot position in every row, the columns of A do <u>not</u> span $\mathbb{R}^3$ $\Leftrightarrow$ the vectors in S do <u>not</u> span $\mathbb{P}_2$.

7. (10 pts) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(a) Find $P_{\mathcal{B}}$, the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^2$.

$$P_{\mathcal{B}} = [b_1 \; b_2] = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$$

(b) Suppose $\mathbf{x} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. Find the coordinates of $\mathbf{x}$ relative to $\mathcal{B}$, i.e. find $[\mathbf{x}]_{\mathcal{B}}$.

<u>Method 1</u> $\quad P_{\mathcal{B}}^{-1} \vec{x} = [\vec{x}]_{\mathcal{B}}$

$$P_{\mathcal{B}}^{-1} = \frac{1}{-2}\begin{pmatrix} 3 & -2 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} -3/2 & 1 \\ 2 & -1 \end{pmatrix}$$

$$P_{\mathcal{B}}^{-1} \vec{x} = \begin{pmatrix} -3/2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ 14 \end{pmatrix}$$

<u>Method 2</u>

$$\left(\begin{array}{cc|c} 2 & 2 & 8 \\ 4 & 3 & 2 \end{array}\right)^{\frac{1}{2}} \sim \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 4 & 3 & 2 \end{array}\right)_{-4R_1} \sim \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -1 & -14 \end{array}\right)^{+R2}$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & -1 & -14 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & 1 & 14 \end{array}\right) \quad \text{So } [x]_{\mathcal{B}} = \begin{pmatrix} -10 \\ 14 \end{pmatrix}$$

<u>check</u> $\quad -10\, b_1 + 14 b_2 \overset{?}{=} \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

$$-10\begin{pmatrix} 2 \\ 4 \end{pmatrix} + 14\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -20+28 \\ -40+42 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} \checkmark$$

8. (10 pts) Let $A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 \end{pmatrix}$.

(a) Find $\det A$.

$$\begin{vmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 \end{vmatrix} = (-2)\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -1 \end{vmatrix} \begin{matrix} \\ -R_1 \\ -R_1 \end{matrix} = (-2)\begin{vmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -2(1)\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

(b) Find $\det(2A^3)$. Since A has 4 rows

$$= 2^4 \left(|A|^3\right) = 16 \cdot (-2)^3 = 16(-8) = -128$$

9. (10 pts) Let $T : \mathbb{R}^4 \to \mathbb{M}_{2\times 3}$ be defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 3a & a+c & 0 \\ 0 & 0 & 2c \end{bmatrix}.$$

Then T is linear. (You do not need to show this.)

(a) Find a basis for the range of T. Show work but no justification of basis properties is needed.

range of T =

$$\left\{ \begin{bmatrix} 3a & a+c & 0 \\ 0 & 0 & 2c \end{bmatrix} = a\begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

Basis: $\left\{ \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right\}$

(b) Find a basis for ker T. Show work but no justification of basis properties is needed.

Find all $x \in \mathbb{R}^4$ such that $T(x) = 0_{M_{2\times 3}}$

Solve: $\begin{bmatrix} 3a & a+c & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

We need $a=0$ and $c=0$

$\ker T = \left\{ \begin{pmatrix} 0 \\ b \\ 0 \\ d \end{pmatrix} : b, d \in \mathbb{R} \right\}$ Basis: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(c) Is T one-to-one? Briefly justify your answer.

Since $\ker T \neq 0_{\mathbb{R}^4}$, T is not 1-1.

10. (8 pts) A linear transformation is called *onto* if for every vector $w \in W$ there is at least one vector $v \in V$ with $T(v) = w$. Suppose that the linear transformation $T : V \to W$ is onto. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a spanning set for V. Show that $S' = \{T(v_1), T(v_2), T(v_3)\}$ is a spanning set for W. The first line of your proof should be: "Let w be any vector in W."

proof Let w be any vector in W. Since T is onto, $\exists$ $v \in V$ with $T(v) = w$. Since the vectors in S span V, $\exists$ scalars c_1, c_2, c_3 with $v = c_1 v_1 + c_2 v_2 + c_3 v_3$.

Then $T(v) = T(c_1 v_1 + c_2 v_2 + c_3 v_3) = w$

$c_1 T(v_1) + c_2 T(v_2) + c_3 T(v_3) = w$ since T is linear

But now we have written w as a linear combination of the vectors in S'. Since w was chosen arbitrarily, this shows the vectors in S' span W. $\boxtimes$