Math 46 Exam II Solutions Fall 2006

1) $\det\begin{pmatrix}1 & 1 & 1\\ -3 & 8 & -4\\ 2 & -3 & 2\end{pmatrix} = \det\begin{pmatrix}1 & 0 & 0\\ -3 & 11 & -1\\ 2 & -5 & 0\end{pmatrix} = \det\begin{pmatrix}11 & -1\\ -5 & 0\end{pmatrix} = -5$

2) $5A = \begin{pmatrix}15 & 5\\ 20 & 10\end{pmatrix}$ $\quad\det 5A = 5^2 \det A$ so

$\det 5A \neq 5 \det A$.

3) $\det(PAP^{-1}) = \det(P)\cdot\det(A)\cdot\det(P^{-1}) =$

$\det P \cdot \det(P^{-1}) \det A = \det(PP^{-1})\cdot\det A =$

$= \det I \cdot \det A = 1\cdot\det A = \det A$

4) $\begin{pmatrix}1 & -2 & 6 & 5 & 0\\ 0 & 1 & -1 & -3 & 3\\ 0 & -1 & 2 & 3 & -1\\ 1 & 1 & -1 & -4 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -2 & 6 & 5 & 0\\ 0 & 1 & -1 & -3 & 3\\ 0 & 0 & 1 & 0 & 2\\ 0 & 3 & -7 & -9 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -2 & 6 & 5 & 0\\ 0 & 1 & -1 & -3 & 3\\ 0 & 0 & 1 & 0 & 2\\ 0 & 0 & -4 & 0 & 8\end{pmatrix}$

Last two rows are proportional so first 3 columns are pivot columns so first three vectors in list form a basis for the span of all 5 vectors

5) a) $T\left[\begin{pmatrix}a & b\\ c & d\end{pmatrix} + \begin{pmatrix}e & f\\ g & h\end{pmatrix}\right] = T\left[\begin{pmatrix}a+e & b+f\\ c+g & d+h\end{pmatrix}\right] =$

$\begin{pmatrix}a+e-(b+f)\\ 2(c+g)\end{pmatrix} = \begin{pmatrix}(a-b)+(e-f)\\ 2c+2g\end{pmatrix} = \begin{pmatrix}a-b\\ 2c\end{pmatrix} + \begin{pmatrix}e-f\\ 2g\end{pmatrix} =$

$T\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] + T\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]$ T preserves sums

$$T\left[k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = T\left[\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}\right] = \begin{pmatrix} ka - kb \\ 2kc \end{pmatrix} =$$

$$\begin{pmatrix} k(a-b) \\ k(2c) \end{pmatrix} = k\begin{pmatrix} a-b \\ 2c \end{pmatrix} = kT\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$$ T preserves scalar mult.

$\therefore$ T is linear

b) $T\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} a-b \\ 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if $a = b$ and $c = 0$

$\therefore$ ker T = All matrices $\begin{pmatrix} a & a \\ 0 & d \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

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6) a) $p(t) = -2(1-t) + 1(1+t) + 3(1+t+t^2) = 2 + 6t + 3t^2$

b) use coordinate vectors: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} \\ +R_1 \\ \\ \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} -R_3 \\ -2R_3 \\ \\ \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} \\ \frac{1}{2} \\ \\ \end{matrix} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{matrix} -R_2 \\ \\ \\ \end{matrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$ so $[t^2+1]_\beta = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ check: $t^2+1 \stackrel{?}{=} \frac{1}{2}(1-t) - \frac{1}{2}(1+t) + 1(1+t+t^2)$

$= \frac{1}{2} - \frac{1}{2}t - \frac{1}{2} - \frac{1}{2}t + 1 + t + t^2$

$= 1 + t^2$ ✓

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7) a) no b) no c) no d) no e) yes

8) a) $\underbrace{\dim \text{Nul } A}_{\geq 6} + \text{rank } A = 14$ so $0 \leq \text{rank } A \leq 8$

(ie. 6-14)

b) $\underbrace{\dim \text{Nul } A} + \text{rank } A = 3$

$0 \leq \dim \text{Nul } A \leq 3$ so minimum is 0

9) Let $T(x) = Ax$ then A is 6×10.

dim ker T = dim Nul A, dim range of T = dim col A

dim ker T + dim range of T = 10

a) Max = 10. Let $T(\underline{x}) = \underline{0}$ for all $x \in \mathbb{R}^{10}$

b) Max = 6. Since A is 6×10, A has at most 6 pivot positions so dim col A ≤ 6

10. Suppose $\bar{u}$ and $\bar{v}$ are in ker T. Then

$T(\bar{u}) = T(\bar{v}) = \bar{0}$. I want to show

$\bar{u} + \bar{v}$ in ker T

$c\bar{u}$ in ker T for any coeff c.

$T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v}) = \bar{0} + \bar{0} = \bar{0}$ So $\bar{u}+\bar{v}$ in ker T

$T(c\bar{u}) = c\,T(\bar{u}) = 0$ $c\bar{u}$ in ker T.