Math 46 Exam II Fall 2008 Solutions

$\boxed{1}$ (10) (i) T (ii) F (need it closed under + and scalar mult.) (iii) F $\vec{0} \notin$ subset ($A\vec{0} = \vec{0} \neq \vec{v} \leftarrow$ nonzero)

(iv) F $(\dim V \leq m)$ (v) T

$\boxed{2}$ (10) (i) $\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 - -2 = 3 \neq 0 \Rightarrow$ cols are LI $\Rightarrow$ Since $\dim \mathbb{R}^2 = 2$, cols span $\mathbb{R}^2$

$\Rightarrow$ cols form a basis for $\mathbb{R}^2$

(ii) $[\vec{v}]_B = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ so $\vec{v}_B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ or $\vec{v} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

(iii) $P_B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$\boxed{3}$ (10) (i) $[\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ (ii) $[\ ]_B : V \to \underline{\underline{\mathbb{R}^n}}$

(iii) Claim: $[\ ]_B : V \to \mathbb{R}^n$ is 1-1.

Let $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ and $\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$ be 2 arbitrary vectors in V.

Suppose $[\vec{u}]_B = [\vec{v}]_B$. Then $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \rightarrow$ $c_1 = d_1$, $c_2 = d_2$, $\vdots$, $c_n = d_n$ $\rightarrow \vec{u} = \vec{v}$ $\boxtimes$

$\boxed{4}$ (6) i) $\{a + bt + ct^2 : b = 0\}$ yes

ii) $\{a + bt + ct^2 : b = c\}$ yes

iii) $\{a + bt + ct^2 : a = 3\}$ No $\vec{0} \notin$ subset

$\boxed{5}$ (14) (i) $\begin{vmatrix} 3 & 1 & 2 \\ 4 & 0 & 1 \\ 9 & 6 & 1 \end{vmatrix} = -4\begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 9 & 6 \end{vmatrix} = -4(-11) - 1(9) = 44 - 9 = 35$

(ii) a) $|B| = -|A| = -4$ b) $|B| = -\frac{1}{3}|A| = -4/3$ c) $|B| = |A| = 4$ d) $|B| = |A|^3 = 64$ e) $|B| = \frac{1}{|A|} = \frac{1}{4}$

$\boxed{6}$ (12) (i) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent if the only solution to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0} \text{ is } c_1 = c_2 = \cdots = c_n = 0.$$

(ii) Since $\vec{v} \in V$ and $\mathcal{B}$ is a basis $\exists$ unique scalars $c_1, c_2, \ldots c_n$ such that $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$. Then $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n - \vec{v} = \vec{0}$ is a dependence relation, so $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n, \vec{v}\}$ is L.D.
(not all scalars are 0)

(iii) Let $\vec{p}_1(t) = 1 + 2t + 3t^2$, $\vec{p}_2(t) = 4 + 5t + 6t^2$, $\vec{p}_3(t) = 2 + t$.

Then $\{[\vec{p}_1(t)]_\beta, [\vec{p}_2(t)]_\beta, [\vec{p}_3(t)]_\beta\} = \left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$.

We determine whether or not these vectors are L.I. in $\mathbb{R}^3$.

Let $A = \begin{bmatrix}1&4&2\\2&5&1\\3&6&0\end{bmatrix}$. Then these columns are LI $\Leftrightarrow$ the only soln to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c}1&4&2&0\\2&5&1&0\\3&6&0&0\end{array}\right]\begin{array}{l}\\-2R_1\\-3R_1\end{array} \sim \left[\begin{array}{ccc|c}1&4&2&0\\0&-3&-3&0\\0&-6&-6&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&4&2&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$

Since A does not have a pivot position in each column, $A\vec{x} = \vec{0}$ has free variables $\Rightarrow$ $A\vec{x} = \vec{0}$ has non-trivial solutions $\Rightarrow$ its columns are L.D. $\Rightarrow \{\vec{p}_1(t), \vec{p}_2(t), \vec{p}_3(t)\}$ is <u>LD</u> due to the isomorphism between $\mathbb{P}_2$ and $\mathbb{R}^3$. $\boxtimes$

7 (12)

$$A = \begin{pmatrix} 1 & -3 & -2 & 12 & -4 \\ 0 & 7 & 7 & -12 & 11 \\ -4 & -7 & -11 & -13 & -9 \\ -1 & 0 & -1 & -7 & -1 \end{pmatrix} \sim \begin{array}{c} \begin{matrix} p & p & & p & \end{matrix} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = B$$

(i) Basis for row $A = \{(1,2,3,4,5), (0,1,1,-2,1), (0,0,0,1,2)\}$ (nonzero rows of B)

(ii) Basis for col A: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 12 \\ -12 \\ -13 \\ -1 \end{bmatrix} \right\}$ (pivot columns of A)

(iii) Basis for nul A: We must reduce B further to RREF.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} -4R_3 \\ +2R_3 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} -2R_2 \sim \begin{array}{c} \begin{matrix} & & f & & f \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & -13 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$x_1 + x_3 - 13x_5 = 0$
$x_2 + x_3 + 5x_5 = 0$
$x_3 = x_3$
$x_4 + 2x_5 = 0$
$x_5 = x_5$

$x_1 = -x_3 + 13x_5$
$x_2 = -x_3 - 5x_5$
$x_3 = x_3$
$x_4 = -2x_5$
$x_5 = x_5$

Basis for nul A = $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ -5 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

(iv) Basis for nul B = Basis for nul A since $A \sim B$. Basis for nul B

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8 (14) A is 4×11

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\vec{x} \mapsto A\vec{x} : \mathbb{R}^{11} \to \mathbb{R}^4$

$n = 11 = \text{rank of } A + \dim \text{Nul } A$

$0 \le \text{rank of } A \le 4$

$7 \le \dim \text{nul } A \le 11$

i) rank $A \le 4$
ii) rank $A \ge 0$
iii) $\dim(\text{nul } A) \le 11$
iv) $\dim(\text{nul } A) \ge 7$

(v) Row A is a subspace of $\mathbb{R}^{11}$
(vi) Col A is a subspace of $\mathbb{R}^4$
(vii) Nul A is a subspace of $\mathbb{R}^{11}$

$\boxed{9}$ (12) $T: \mathcal{V} \to \mathcal{W}$

(i) $\ker T = \{ \vec{x} \in \mathcal{V} \mid T(\vec{x}) = \vec{0}_W \}$

(ii) range of $T = \{ T(\vec{x}) \mid \vec{x} \in \mathcal{V} \}$

(iii) Ker T is a subspace of $\mathcal{V}$

(a) $\vec{0}_V \in \ker T$ because T linear $\to$ $T(\vec{0}_V) = \vec{0}_W$

(b) Ker T is closed under + :

Let $\vec{u}, \vec{v} \in \ker T$ Then $T(\vec{u}) = \vec{0}_W$ and $T(\vec{v}) = \vec{0}_W$.

We need to show $\vec{u} + \vec{v} \in \ker T$.

$$T(\vec{u} + \vec{v}) \underset{\substack{\uparrow \\ \text{T is linear}}}{=} T(\vec{u}) + T(\vec{v}) = \vec{0}_W + \vec{0}_W = \vec{0}_W \quad \text{so} \quad \vec{u} + \vec{v} \in \ker T.$$

(c) Ker T is closed under scalar multiplication.

Let $\vec{u} \in \ker T$, c a scalar. Then $T(\vec{u}) = \vec{0}_W$.

We need to show $c\vec{u} \in \ker T$.

$$T(c\vec{u}) \underset{\substack{\uparrow \\ \text{T is linear}}}{=} cT(\vec{u}) = c \cdot \vec{0}_W = \vec{0}_W \quad \text{so} \quad c\vec{u} \in \ker T$$

$\therefore$ Ker T is a subspace of $\mathcal{V}$.