Math 70
Linear Algebra

TUFTS UNIVERSITY
Department of Mathematics
Exam 2

April 1, 2013
All sections

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	10	
2	12	
3	12	
4	10	
5	5	
6	6	
7	9	
8	8	
9	10	
10	10	
11	8	
	100	

1. (10 pts) **True/false questions.** For each of the statements below, decide whether it is true or false. Indicate your answer by shading the corresponding box. There will be no partial credit.

   (a) Consider a system of n equations in n variables. If the matrix of coefficients has rank n, the solution is unique. [T] [F]

   (b) It is possible to have a set of vectors in $\mathbb{R}^2$ that spans $\mathbb{R}^2$ but which is not linearly independent. [T] [F]

   (c) $\text{Span}\{v_1, v_2\}$ is the same space as $\text{Span}\{v_1, 2v_2\}$ [T] [F]

   (d) If the dimension of the column space of A is n, then A cannot have more than n rows. [T] [F]

   (e) If matrices A and B are of the same size and k is a scalar, then $\text{rank}(A) + \text{rank}(B) = \text{rank}(A + B)$ and $\text{rank}(kA) = k \,\text{rank}\, A$. [T] [F]

2. (12 pts) If $A = \begin{bmatrix} 1 & 5 & 0 & 3 & 1 \\ -2 & -10 & 1 & -4 & -3 \\ 3 & 15 & -1 & 7 & 5 \\ 2 & 10 & 0 & 6 & 3 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 5 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is row equivalent to A.

(You do not need to check this). Use this information to determine the following:

(a) A basis for nul(A).

(b) A basis for col(A).

(c) A basis for row(A).

(d) The dimension of col(A) is ______.

(e) The rank of A is ______.

(f) The dimension of nul(A) is ______.

3. (12 pts) Define $T : \mathbb{M}_{2\times 2} \to \mathbb{M}_{2\times 2}$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) Describe the set of all matrices with $T(M) = 0$ (that is, the kernel of T).

(b) Describe the set of all output matrices $T(M)$ (that is, the range of T).

(c) Is T one-to-one? Explain.

(d) Is T onto? Explain.

4. (10 pts) Let $V = \mathbb{M}_{2\times 2}$. The standard basis for V is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Let A, C be two elements of this vector space defined as $A = \begin{bmatrix} 3 & -6 \\ 4 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Find $[A]_{\mathcal{B}}$ and $[C]_{\mathcal{B}}$.

(b) Is $\{A, C\}$ a linearly independent set in V? Justify your answer.

(c) Is $\mathbb{M}_{2\times 2}$ isomorphic to $\mathbb{R}^4$? Explain your answer.

5. (5 pts)To show that in general, $\det(A+B) \neq \det(A) + \det(B)$, it suffices to give a an example for which equality does not hold. Give one example of a pair of 2×2 matrices A, B such that this equality does not hold.

6. (6 points) Find

$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & 2 & -2 & 4 \\ 2 & 4 & 1 & 0 \end{vmatrix}$$

7. (9 pts) Determine whether each of the following sets of $\mathbb{R}^3$ is a subspace. No explanation needed. No partial credit.

(a) $W_1 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a = 8b \right\}$. Yes No

(b) $W_2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a = 5 - b \right\}$. Yes No

(c) $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a = \sqrt{b} \right\}$. Yes No

8. (8 pts) Let A be a 4×4 matrix. Given $\det(A) = 4$, fill in the blanks.

(a) $\det(A^T) =$ ______

(b) $\det(A^{-1}) =$ ______

(c) $\det(A^3) =$ ______

(d) $\det(5A) =$ ______

9. (10 pts) Let V be a vector space and let us consider $u_1, u_2, u_3 \in V$. Prove, using the definition of what it means to be a subspace, that $W = \text{Span}\{u_1, u_2, u_3\}$ is a subspace of V.

10. (10 pts) Consider the polynomials $p_1(t) = 1 + t$, $p_2(t) = 4$, $p_3(t) = 1 - t$.

(a) Find a nontrivial solution to

$$c_1p_1(t) + c_2p_2(t) + c_3p_3(t) = \mathbf{0}.$$

(Hint: This can be done by inspection.)

(b) Find a basis for Span$\{p_1, p_2, p_3\}$.

11. (8 pts) Let A be a 10×7 matrix with linearly independent columns. Let B be a 7×7 matrix and assume that dim(Nul(B)) =2.

(a) Prove that dim(Nul(AB)) = 2.

(b) What is dim(col(AB)) and why?

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Name ______________________________

Instructor ______________________________

I pledge that I have neither given nor received assistance on this exam.

Signature ______________________________