Math 70	TUFTS UNIVERSITY	Feb 19, 2020
Linear Algebra	**Department of Mathematics**	**All sections 1,2,3,4,5**
		Exam 1

Instructions: No notes or books are allowed. All calculators, cell phones, or other electronic devices **must** be turned off and put away during the exam. Unless otherwise stated, you **must show all work** to receive full credit. *You are required to sign your exam. With your signature you are pledging that you have neither given nor received assistance on the exam. Students found violating this pledge will receive an F in the course.*

Problem	Point Value	Points
1	12	
2	12	
3	12	
4	13	
5	12	
6	10	
7	10	
8	12	
9	7	
	100	

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1. (12 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 \end{bmatrix}$.

(a) Find all solutions to $Ax = 0$ and write them in parametric vector form.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}.$$

(b) Let $b = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$. Determine if $q = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a solution to $Ax = b$.

$$A\vec{q} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \vec{b},$$

So $\vec{q}$ is <u>not</u> a solution to $A\vec{x} = \vec{b}$.

(c) Again, take $b = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$. Given that $p = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is a particular solution to $Ax = b$, write out the general solution to $Ax = b$ in parametric vector form.

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}$$

2. (12 points)

(a) Complete the definition of a linear transformation T. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if for all u and v in $\mathbb{R}^n$ and all $c \in \mathbb{R}$, ____

$T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ and

$T(c\vec{u}) = cT(\vec{u})$

(b) Let $v_1, ..., v_p \in \mathbb{R}^n$. Write the definition of Span$\{v_1, ..., v_p\}$.

The span of $\{\vec{v}_1, ..., \vec{v}_p\}$ is the set of all linear combinations $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$ of the vectors $\vec{v}_1, ..., \vec{v}_p$.

(c) Let $v_1, ..., v_p \in \mathbb{R}^n$. Define what it means for the set $\{v_1, ...v_p\}$ to be linearly independent.

If $A = [\vec{v}_1 \cdots \vec{v}_p]$, then the set of vectors $\{\vec{v}_1, ..., \vec{v}_p\}$ is linearly independent if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.

3. (12 points) Indicate by shading the appropriate box whether each statement is true or false. *For this problem you do not need to give reasons.*

Let A be a 4×6 matrix and assume that $Ax = b$ is consistent for all $b \in \mathbb{R}^4$.

(a) The columns of A are linearly independent. T F

(b) There exists a nonzero vector x whose image under the linear transformation $T(x) = Ax$ is 0. T F

(c) The linear transformation $T(x) = Ax$ is onto. T F

Let B be a 6×4 matrix and assume that for some $b \in \mathbb{R}^6$, $Bx = b$ has a unique solution.

(d) There are infinitely many solutions to $Bx = 0$. T F

(e) $Bx = b$ is consistent for all $b \in \mathbb{R}^6$. T F

(f) The columns of B span $\mathbb{R}^6$. T F

4. (13 points) Consider a linear system

$$x_1 + \quad x_2 + \quad x_3 = k$$
$$2x_1 + hx_2 + 2x_3 = 3.$$

(a) Write the augmented matrix of the following linear system and row reduce it to row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & k \\ 2 & h & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & k \\ 0 & h-2 & 0 & 3-2k \end{array}\right]$$

(b) Find all values of h and k (if there are any) so the linear system above has

(i) no solutions

$h = 2$, and $3 - 2k \neq 0$

$2k \neq 3$

$k \neq 3/2$

(ii) only one solution

Not possible

(iii) infinitely many solutions.

$h = 2$ and $k = 3/2$

or $h \neq 2$ and k anything

5. (12 points) Consider the following vectors and the sets of vectors.

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(i) $\{u_1\}$ (ii) $\{u_1, u_2\}$ (iii) $\{u_3, u_4\}$

(iv) $\{u_2, u_3, u_4\}$ (v) $\{u_3, u_4, u_5\}$ (vi) $\{u_2, u_3, u_4, u_5\}$

For this problem you do not need to give reasons.

(a) From the above six sets, identify **all** sets that are linearly independent.

(iii), (iv), (v)

(b) From the above six sets, identify **all** sets that span a line in $\mathbb{R}^3$.

(ii)

(c) From the above six sets, identify **all** sets that span a plane in $\mathbb{R}^3$.

(iii)

(d) From the above six sets, identify **all** sets that span the entire $\mathbb{R}^3$.

(iv), (v), (vi)

6. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ 5x_1x_2 \end{bmatrix}.$$

Decide whether T is linear justify your answer in the following way.
• If you believe T is linear, prove it using the definition of linear transformation.
• If you believe T is not linear, provide a **specific counterexample** (i.e., using specific numbers) to one of the conditions in the **definition** of linear transformation.

T is <u>not</u> linear.

If T were linear, for all $\vec{u}, \vec{v} \in \mathbb{R}^2$ and all $c \in \mathbb{R}$, we would have

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

2) $T(c\vec{u}) = cT(\vec{u})$

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $c = 3$.

$T(c\vec{u}) = T\left(\begin{bmatrix} 3 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 90 \end{bmatrix}$

$cT(\vec{u}) = 3T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 3\begin{bmatrix} 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 \\ 30 \end{bmatrix}$

Since $T(c\vec{u}) \neq cT(\vec{u})$,

T is <u>not</u> linear.

7. (10 points) Let $S = \{v_1, v_2, v_3\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Prove that $S' = \{v_1 + v_2, v_2, v_2 + v_3\}$ is also linearly independent.

Suppose there are weights c_1, c_2, c_3 such that

$$c_1(\vec{v}_1 + \vec{v}_2) + c_2\vec{v}_2 + c_3(\vec{v}_2 + \vec{v}_3) = \vec{0}.$$

Rearranging, we get

$$c_1\vec{v}_1 + (c_1 + c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = 0.$$

Since $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent,

$c_1 = 0$, $c_1 + c_2 + c_3 = 0$, and $c_3 = 0$.

These combine to tell us that

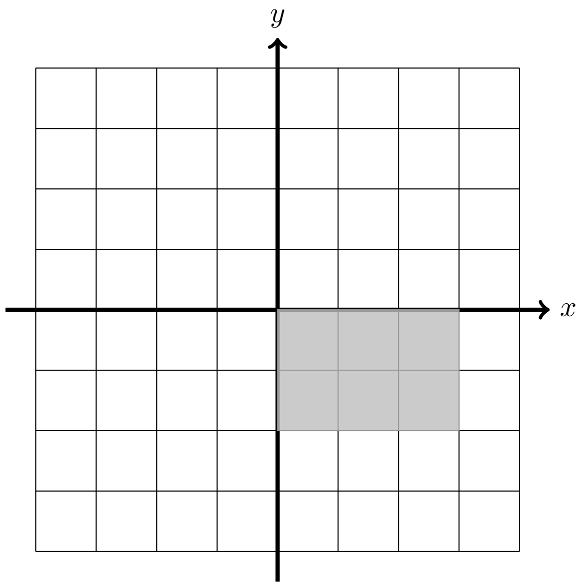
$c_2 = 0$ as well.

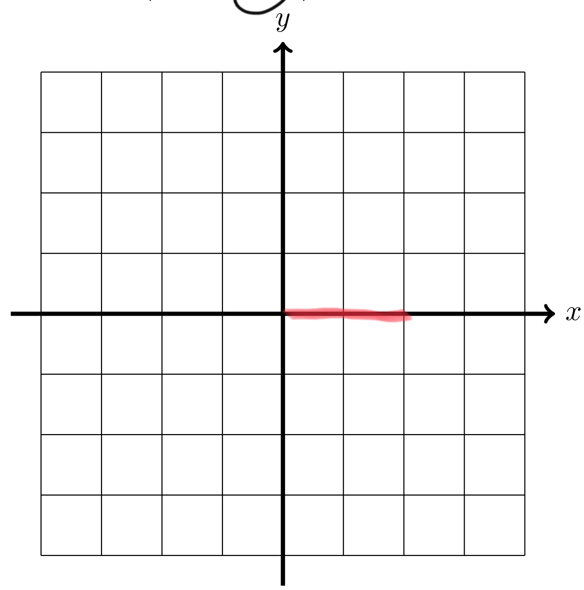
Thus S' is also linearly independent.

8. (12 points)

(a) Consider the standard matrix of a linear transformation: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

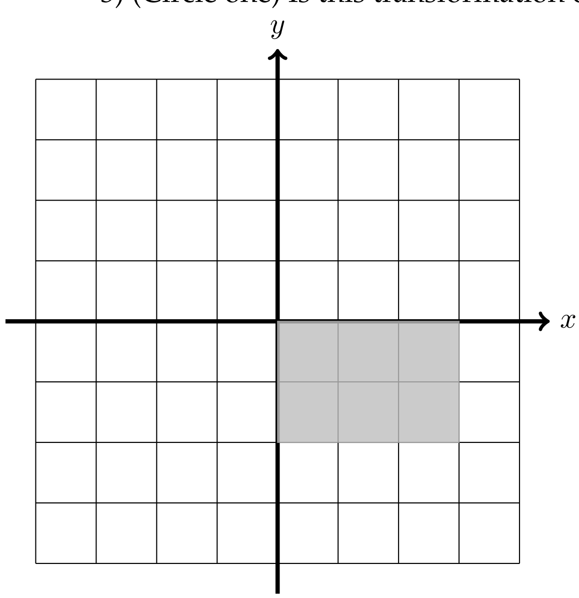
1) Draw the image of the figure below under the transformation.
2) (Circle one) Is this transformation one-to-one? (Yes, No)
3) (Circle one) Is this transformation onto? (Yes, No)

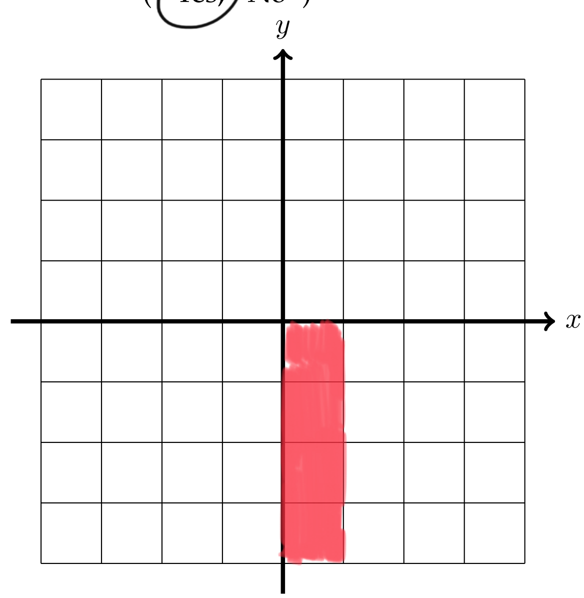




(b) Consider the standard matrix of a linear transformation: $B = \begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix}$.

1) Draw the image of the figure below under the transformation.
2) (Circle one) Is this transformation one-to-one? (Yes, No)
3) (Circle one) Is this transformation onto? (Yes, No)



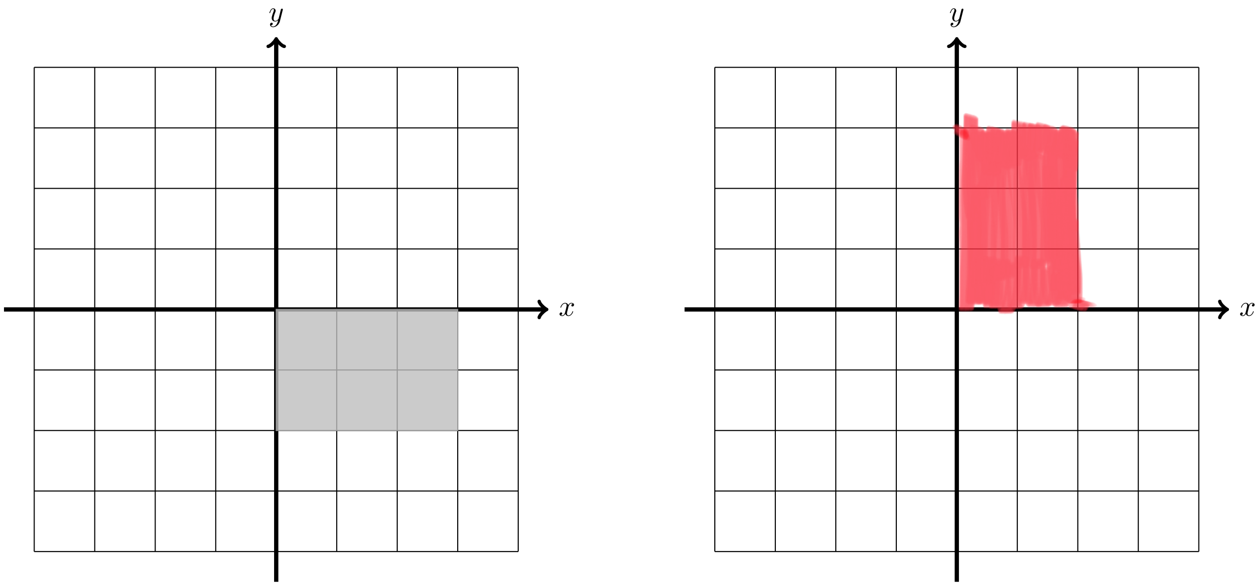


(c) Consider the standard matrix of a linear transformation: $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

1) Draw the image of the figure below under the transformation.

2) (Circle one) Is this transformation one-to-one? (Yes, No)

3) (Circle one) Is this transformation onto? (Yes, No)



9. (7 points) Find the standard matrix of $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_2 + x_3 \\ -x_3 \\ x_1 + x_2 + 7x_3 \end{bmatrix}.$$

$$A = \left[T(\vec{e}_1) \ T(\vec{e}_2) \ T(\vec{e}_3) \right]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 7 \end{bmatrix}$$

Math 70 **Exam I** **February 19, 2020**

Name: Solutions

Please circle your section

Section 1, Eunice Kim, TTh 10:30–11:45

Section 2, Genevieve Walsh, TTh 1:30–2:45

Section 3, Yanghui Liu, MW 9:00–10:15

Section 4, Curtis Heberle, TTh 12:00–1:15

Section 5, Nathan Fisher, MW 9:00–10:15

I pledge that I have neither given nor received assistance on this exam.

Signature ______________________________